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Tableaux for multi-modal hybrid logic with binders, transitive relations and relation hierarchies

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ABSTRACT

In a previous paper, a tableau calculus has been presented, which constitute a decision procedure for hybrid logic with the converse and global modalities and a restricted use of the binder. This work extends such a calculus to multi-modal logic with transitive relations and relation inclusion assertions.

The separate addition of either transitive relations or relation hierarchies to the considered decidable fragment of multi-modal hybrid logic can easily be shown to stay decidable, by resorting to results already proved in the literature. However, such results do not directly allow for concluding whether the logic including both features is still decidable. The existence of a terminating, sound and complete calculus for the considered logic proves that the addition of transitive relations and relation hierarchies to such an expressive decidable fragment of hybrid logic yields a decidable logic.

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1 Introduction

Hybrid languages are extensions of modal logic that allow for naming and accessing states of a structure explicitly. Their main distinguishing feature is represented by special atomic propositions, called *nominals*, which give names to states: a nominal is true in exactly one state of the model. The two operators specific of hybrid languages are the *satisfaction operator* (@), allowing for jumping to a point named by a nominal, regardless of the accessibilities in the structure, and the *binder* ($\downarrow$), allowing for binding *state variables* to states dynamically and for referring to these states later on.

Other modal operators can be added to the basic hybrid language. Among them, this work considers the converse modalities ($\Diamond^-$ and $\Box^-$) and the global ones (E and A). Moreover, a hybrid language can rely on a multi-modal base, allowing for modelling structures with different accessibility relations. In this case, the basic modalities $\Diamond$ and $\Box$ (and their converses, if present) are indexed by relation symbols. Hybrid multi-modal languages can also be enriched with a feature largely used in description logics, i.e. the possibility of declaring an accessibility relation to be transitive and/or included in another one.

In this paper, basic hybrid logic (with nominals only, beyond the modal operators $\Diamond$ and $\Box$) will be denoted by HL, and basic multi-modal hybrid logic by HL_m. Logics extending HL or HL_m with operators $O_1, \ldots, O_n$ are denoted by $\mathsf{HL}(O_1, \ldots, O_n)$ and $\mathsf{HL}_m(O_1, \ldots, O_n)$, respectively. A multi-modal language including transitivity assertions and/or relation hierarchies will be denoted in the same way, just including Trans (for transitivity) and/or $\sqsubseteq$ (for relation inclusion) among $O_1, \ldots, O_n$.

The satisfiability problem for formulae of any hybrid logic $\mathsf{HL}(O_1, \ldots, O_n)$ or $\mathsf{HL}_m(O_1, \ldots, O_n)$ – where the operators are among those considered above – is decidable, provided that the binder is not among $O_1, \ldots, O_n$. Unfortunately, due to its high expressive power, the addition of the binder causes a loss of decidability (even in the absence of nominals) [1, 2].

However, there are both syntactic and semantic restrictions allowing for regaining decidability of hybrid logic with the binder. In [20] it is proved that the satisfiability problem for formulae in $\mathsf{HL}(@, \downarrow, \mathsf{E}, \Diamond^-)$ is decidable, provided that their negation normal form contains no universal operator (i.e. either $\Box$ or $\Box^-$ or A) scoping over a binder, that in turn has scope over a universal operator. Such a fragment of hybrid logic is denoted by $\mathsf{HL}(@, \downarrow, \mathsf{E}, \Diamond^-) \setminus \Box\!\downarrow\!\Box$. Another way of restoring decidability for hybrid logic with binders is by restricting the frame class: for instance, decidability of $\mathsf{HL}(@, \downarrow, \mathsf{E}, \Diamond^-)$ can be recovered by restricting the frame class to frames of bounded width (i.e. frames where the number of successors of each state is bounded) [20]. Moreover, $\mathsf{HL}(\downarrow)$ on transitive frames, transitive trees, linear orders or equivalence relations is decidable [9, 17, 18, 15].

On the other hand, adding either the satisfaction operator or the converse modalities to the binder leads to undecidability over transitive frames [18], and satisfiability becomes undecidable also for the multi-modal version of the hybrid language: $\mathsf{HL}_m(\downarrow)$ is undecidable over a wide range of frame classes, among which the classes of frames mentioned above [16] .

This work considers the fragment of hybrid logic $\mathsf{HL}_m(@, \downarrow, \mathsf{E}, \Diamond^-, \mathsf{Trans}, \sqsubseteq) \setminus \Box\!\downarrow\!\Box$, i.e. hybrid logic with the converse and global modalities, transitive relations, relation hierarchies and restricted uses of the binder. The results quickly reported above do not allow for concluding whether such an extension of $\mathsf{HL}(@, \downarrow, \mathsf{E}, \Diamond^-) \setminus \Box\!\downarrow\!\Box$ is decidable. Nor can the question be answered by the reduction argument used in [20] to prove that $\mathsf{HL}(@, \downarrow, \mathsf{E}, \Diamond^-) \setminus \Box\!\downarrow\!\Box$ is decidable. The proof in that work shows that there exists a satisfiability preserving translation of formulae in $\mathsf{HL}(@, \downarrow, \mathsf{E}, \Diamond^-) \setminus \Box\!\downarrow\!\Box$ into $\mathsf{HL}(@, \downarrow, \mathsf{E}, \Diamond^-) \setminus \downarrow\!\Box$, i.e. the set of formulae in negation normal form where no universal operator occurs in the scope of a binder. The standard translation of hybrid logic into first order classical logic [1, 20] maps, in turn, formulae in $\mathsf{HL}(@, \downarrow, \mathsf{E}, \Diamond^-) \setminus \downarrow\!\Box$ into universally guarded formulae, that have a decidable satisfiability problem [10]. Since also the translation of a relation inclusion axiom is a guarded formula, $\mathsf{HL}_m(@, \downarrow, \mathsf{E}, \Diamond^-, \sqsubseteq) \setminus \Box\!\downarrow\!\Box$ is decidable. But transitivity axioms make the guarded fragment (GF) of first order logic undecidable [10].

On the other side, if transitive relations only occur in guards, GF is decidable [19], therefore $\mathsf{HL}_m(@, \downarrow, \mathsf{E}, \Diamond^-, \mathsf{Trans}) \setminus \Box\!\downarrow\!\Box$ is decidable. But the translation of relation inclusion axioms may have transitive relations outside guards. Therefore, in the presence of both transitive relations and relation hierarchies, the decidability question is unsettled.

This work constitutes a prosecution of previous works, where terminating tableau calculi for decidable fragments of Hybrid Logic with the binder have been defined [6, 7]. In particular, in [7], a tableau calculus has been presented, which constitutes a satisfiability decision procedure for $\mathsf{HL}(@, \downarrow, \mathsf{E}, \Diamond^-) \setminus \Box\!\downarrow\!\Box$. Such a procedure is here extended to multi-modal hybrid logic $\mathsf{HL}_m(@, \downarrow, \mathsf{E}, \Diamond^-, \mathsf{Trans}, \sqsubseteq) \setminus \Box\!\downarrow\!\Box$: a tableau calculus extending that given in [7] is defined, which terminates and is sound and complete for formulae in the fragment $\mathsf{HL}_m(@,\downarrow,\mathsf{E},\Diamond^-,\mathsf{Trans},\sqsubseteq)\setminus\downarrow\Box$, i.e. formulae in negation normal form where no occurrence of a universal operator is in the scope of a binder, with the addition of transitivity assertions and relation hierarchies. A satisfiability preserving translation of formulae, similar to that defined in [20], turns the calculus into a satisfiability decision procedure for the fragment $\mathsf{HL}_m(@,\downarrow,\mathsf{E},\Diamond^-,\mathsf{Trans},\sqsubseteq)\setminus\Box\downarrow\Box$.

Soundness, completeness and termination of the tableaux calculus thus imply that the satisfiability problem for the fragment of multi-modal hybrid logic $\mathsf{HL}_m(@,\downarrow,\mathsf{E},\Diamond^-,\mathsf{Trans},\sqsubseteq)\setminus\Box\downarrow\Box$ is decidable.

The language $\mathsf{HL}_m(@,\downarrow,\mathsf{E},\Diamond^-,\mathsf{Trans},\sqsubseteq)\setminus\Box\downarrow\Box$ allows for representing some interesting frame properties. For instance, assuming that a transitivity assertion has the form $\mathsf{Trans}(R)$, where R is a relation symbol, and inclusion assertions may have either form $S\sqsubseteq R$ (S is a sub-relation of R) or $S^-\sqsubseteq R$ (the inverse of S is a sub-relation of R), the following properties can be represented:

Transitivity:	$\mathsf{Trans}(R)$
Symmetry:	$R^-\sqsubseteq R$
Reflexivity:	$\mathsf{A}\downarrow x.\Diamond_R x$
At most n states:	$\mathsf{E}\downarrow x_1.\dots\mathsf{E}\downarrow x_n.\mathsf{A}(x_1\vee\cdots\vee x_n)$
At least one R-sibling:	$\mathsf{A}\downarrow x.\Diamond_R^-\Diamond_R\neg x$
At least n R-successors:	$\mathsf{A}\downarrow x.\Diamond\downarrow y_1.(x:\Diamond_R(\neg y_1\wedge\downarrow y_2.(x:\Diamond_R(\neg y_1\wedge\neg y_2\wedge\downarrow y_3.\dots))))$

Therefore satisfiability in $\mathsf{HL}_m(@,\downarrow,\mathsf{E},\Diamond^-)\setminus\Box\downarrow\Box$ is decidable over frames enjoying any combination of the above properties.

The tableau calculus is presented in Section 3. Section 4 proposes a variant of the set of expansion rules specific to treat transitive relations and relation inclusion assertions, that is closer to the corresponding one proposed, for instance, in [11, 12, 13, 14]. Section 5 shows how to extend the termination and completeness proofs of [7] to the extended calculus and Section 6 concludes this work.

2 Syntax and semantics of multi-modal hybrid logic with transitive relations and inclusion assertions

Well-formed expressions of $\mathsf{HL}_m(@,\downarrow,\mathsf{E},\Diamond^-,\mathsf{Trans},\sqsubseteq)$ are partitioned into two categories: *formulae* (for which the metasymbols F,G,H – possibly with subscripts – will be used) and *assertions* (that will be denoted by A,B,C).

Formulae are built out of a set PROP of propositional letters, a set NOM of nominals, an infinite set VAR of state variables, and a set REL of relation symbols (all such sets being mutually disjoint), and defined by the following grammar:

$$\begin{array}{rl} F := & p \mid a \mid x \mid \neg F \mid F\wedge F \mid F\vee F \mid \Diamond_R F \mid \Box_R F \\ & \mid \Diamond_R^- F \mid \Box_R^- F \mid \mathsf{E}F \mid \mathsf{A}F \mid a{:}F \mid x{:}F \mid \downarrow x.F \end{array}$$

where $p\in\mathsf{PROP}$, $a\in\mathsf{NOM}$, $x\in\mathsf{VAR}$ and $R\in\mathsf{REL}$. In this work, the notation $t{:}F$ is used (for $t\in\mathsf{NOM}\cup\mathsf{VAR}$) rather than $@_tF$. We use metavariables a,b,c,d for nominals, while x,y,z are used for state variables and R,S,P for relation symbols (every metavariable possibly decorated by a subscript).

If F is a formula, x a state variable and a a nominal, then $F[a/x]$ denotes the formula obtained from F by replacing a for every free occurrence of x (an occurrence of x is free if it is not in the scope of a $\downarrow x$). If $a_0,\dots,a_n,b_0,\dots,b_n$ are nominals, then $F[b_0/a_0,\dots,b_n/a_n]$ denotes the formula obtained from F by simultaneously replacing b_i for every occurrence of a_i.

Assertions are either *transitivity assertions,* of the form $\mathsf{Trans}(R)$, for $R\in\mathsf{REL}$, or *inclusion assertions,* of either form $R\sqsubseteq S$ or $R^-\sqsubseteq S$, for $R,S\in\mathsf{REL}$. Here, R^- is intended to denote the inverse of the relation denoted by R, i.e. the set of pairs of states $\langle w,w'\rangle$ such that $\langle w',w\rangle$ is in the relation denoted by R.

Note that inverse relations are allowed only on the left of the $\sqsubseteq$ symbol. This is only a syntactical restriction, since $R^-\sqsubseteq S^-$ is equivalent to $R\sqsubseteq S$, and $R\sqsubseteq S^-$ is equivalent to $R^-\sqsubseteq S$.

An *interpretation* $\mathcal{M}$ of an $\mathsf{HL}_m(@,\downarrow,\mathsf{E},\Diamond^-,\mathsf{Trans},\sqsubseteq)$ language is a tuple $\langle W,\rho,N,I\rangle$ where W is a non-empty set (whose elements are the *states* of the interpretation), ρ is a function mapping every $R\in\mathsf{REL}$ to a binary relation on W ($\rho(R)\subseteq W\times W$), N is a function $\mathsf{NOM}\to W$ and I a function $W\to 2^{\mathsf{PROP}}$. We shall write wRw' as a shorthand for $\langle w,w'\rangle\in\rho(R)$, and wR^-w' for $\langle w',w\rangle\in\rho(R)$.

If $\mathcal{M}=\langle W,\rho,N,I\rangle$ is an interpretation, $w\in W$, σ is a variable assignment for $\mathcal{M}$ (i.e. a function $\mathsf{VAR}\to W$) and F is a formula, the relation $\mathcal{M}_w,\sigma\models F$ is defined adding the following clauses to the usual definition of the classical operators:

1. $\mathcal{M}_w, \sigma \models p$ if $p \in I(w)$, for $p \in \mathsf{PROP}$.
2. $\mathcal{M}_w, \sigma \models a$ if $N(a) = w$, for $a \in \mathsf{NOM}$.
3. $\mathcal{M}_w, \sigma \models x$ if $\sigma(x) = w$, for $x \in \mathsf{VAR}$.
4. $\mathcal{M}_w, \sigma \models a{:}F$ if $\mathcal{M}_{N(a)}, \sigma \models F$, for $a \in \mathsf{NOM}$.
5. $\mathcal{M}_w, \sigma \models x{:}F$ if $\mathcal{M}_{\sigma(x)}, \sigma \models F$, for $x \in \mathsf{VAR}$.
6. $\mathcal{M}_w, \sigma \models {\downarrow}x.F$ if $\mathcal{M}_w, \sigma_x^w \models F$, where σ_x^w is the variable assignment such that $\sigma_x^w(x) = w$ and, for $y \neq x$, $\sigma_x^w(y) = \sigma(y)$.
7. $\mathcal{M}_w, \sigma \models \Box_R F$ if for every w' such that wRw', $\mathcal{M}_{w'}, \sigma \models F$.
8. $\mathcal{M}_w, \sigma \models \Diamond_R F$ if there exists w' such that wRw' and $\mathcal{M}_{w'}, \sigma \models F$.
9. $\mathcal{M}_w, \sigma \models \Box_R^- F$ if for every w' such that $w'Rw$, $\mathcal{M}_{w'}, \sigma \models F$.
10. $\mathcal{M}_w, \sigma \models \Diamond_R^- F$ if there exists w' such that $w'Rw$ and $\mathcal{M}_{w'}, \sigma \models F$.
11. $\mathcal{M}_w, \sigma \models \mathsf{A}F$ if $\mathcal{M}_{w'}, \sigma \models F$ for all $w' \in W$.
12. $\mathcal{M}_w, \sigma \models \mathsf{E}F$ if $\mathcal{M}_{w'}, \sigma \models F$ for some $w' \in W$.

A formula F is *satisfiable* if there exist an interpretation $\mathcal{M}$, a variable assignment σ for $\mathcal{M}$ and a state w of $\mathcal{M}$, such that $\mathcal{M}_w, \sigma \models F$. Two formulae F and G are logically equivalent ($F \equiv G$) when, for every interpretation $\mathcal{M}$, assignment σ and state w of $\mathcal{M}$, $\mathcal{M}_w, \sigma \models F$ if and only if $\mathcal{M}_w, \sigma \models G$. A formula F holds in a state w of a model $\mathcal{M}$ ($\mathcal{M}_w \models F$) iff $\mathcal{M}_w, \sigma \models F$ for every variable assignment σ.

Every formula in $\mathsf{HL}_m(@, \downarrow, \mathsf{E}, \Diamond^-)$ is logically equivalent to a formula in negation normal form (NNF), where negation appears only in front of atoms. Therefore, considering only formulae in NNF does not restrict the expressive power of the language.

If $\mathcal{A}$ is a set of assertions, an interpretation $\mathcal{M} = \langle \mathcal{W}, \rho, \mathcal{N}, \mathcal{I} \rangle$ is a model of $\mathcal{A}$ if:

1. for all $R \in \mathsf{REL}$ such that $\mathsf{Trans}(R) \in \mathcal{A}$, and for all states $w_0, w_1, w_2 \in W$, if $w_0 R w_1$ and $w_1 R w_2$, then $w_0 R w_2$;
2. for all $R, S \in \mathsf{REL}$, if $R \sqsubseteq S \in \mathcal{A}$, then $\rho(R) \subseteq \rho(S)$.
3. for all $R, S \in \mathsf{REL}$ and all $w, w' \in W$, if $R^- \sqsubseteq S \in \mathcal{A}$ and wRw', then $w'Sw$.

Finally, if F is a formula and $\mathcal{A}$ a set of assertions, $\{F\} \cup \mathcal{A}$ is satisfiable if there exists a model $\mathcal{M}$ of $\mathcal{A}$ and a state w of $\mathcal{M}$ such that $\mathcal{M}_w \models F$.

3 The tableau calculus

This section shows how to extend the system described in [7] to the presence of transitivity and inclusion assertions. The expansion rules that will be introduced to treat assertions are similar to the analogous rules presented by [11, 12, 13, 14] (see Section 4.1). However, their addition to a terminating calculus dealing also with (restricted occurrences of) the binder is a novelty.

The presentation will be as self contained as possible, therefore it overlaps with the description given in [7] in many points. However, since some of the basic notions underlying the calculus are quite involved, they are not given a completely formal account. We refer to [7] for a more detailed view of the original calculus, while here focusing on the novelties (w.r.t. the system described in [7]) due to the addition of assertions.

$$\frac{}{R \sqsubseteq R}\ \mathsf{Rel}_0$$

$$\frac{R \sqsubseteq S \quad S \sqsubseteq P}{R \sqsubseteq P}\ \mathsf{Rel}_1 \qquad \frac{R^- \sqsubseteq S \quad S \sqsubseteq P}{R^- \sqsubseteq P}\ \mathsf{Rel}_2$$

$$\frac{R^- \sqsubseteq S \quad S^- \sqsubseteq P}{R \sqsubseteq P}\ \mathsf{Rel}_3 \qquad \frac{R \sqsubseteq S \quad S^- \sqsubseteq P}{R^- \sqsubseteq P}\ \mathsf{Rel}_4$$

Table 1: Assertion rules

3.1 The expansion rules

A tableau is a set of branches, and a *tableau branch* is a sequence of *nodes* $n_0, n_1, \ldots$, where each node is labelled either by an assertion or a ground *satisfaction statement*, i.e. a formula of the form $a{:}\,F$, where no state variable occurs free in F. Node labels are always formulae in NNF.

If n occurs before m in a branch, we write $n < m$. The label of the node n is denoted by label(n). The notation $(n)\, a{:}\,F$ is used to denote the node n, and simultaneously say that its label is $a{:}\,F$. If a node $(n)\, a : F$ is in a branch, then the nominal a is said to label the formula F in the branch.

Let F be a ground hybrid formula in NNF and $\mathcal{A}$ a set of assertions. A tableau for $\{F\} \cup \mathcal{A}$ is initialized with a single branch, constituted by the node $(n_0)\, a_0{:}\,F$, where a_0 is a new nominal, followed by nodes labelled by the assertions in $\mathcal{A}$ and then expanded according to the rules of Table 1. Such rules complete the subrelation assertions in $\mathcal{A}$ by the reflexive and transitive closure of $\sqsubseteq$. The formula $a_0{:}\,F$ is the *initial formula* of the tableau.

$$\frac{(n)\, a{:}\,(F \wedge G)}{\begin{array}{c}(m_0)\, a{:}\,F\\(m_1)\, a{:}\,G\end{array}}\ (\wedge) \qquad \frac{(n)\, a{:}\,(F \vee G)}{(m_0)\, a{:}\,F \quad \mid \quad (m_1)\, a{:}\,G}\ (\vee)$$

$$\frac{(n)\, a{:}\,b{:}\,F}{(m)\, b{:}\,F}\ (@) \qquad \frac{(n)\, a{:}\downarrow x.F}{(m)\, a{:}\,F[a/x]}\ (\downarrow)$$

$$\frac{(n)\, a{:}\,\Box_R F \quad (m)\, a{:}\,\Diamond_R b}{(k)\, b{:}\,F}\ (\Box) \qquad \frac{(n)\, a{:}\,\Box_R^- F \quad (m)\, b{:}\,\Diamond_R a}{(k)\, b{:}\,F}\ (\Box^-)$$

$$\frac{(n)\, a{:}\,\Diamond_R F}{\begin{array}{c}(m_0)\, a{:}\,\Diamond_R b\\(m_1)\, b{:}\,F\end{array}}\ (\Diamond) \qquad \frac{(n)\, a{:}\,\Diamond_R^- F}{\begin{array}{c}(m_0)\, b{:}\,\Diamond_R a\\(m_1)\, b{:}\,F\end{array}}\ (\Diamond^-)$$

$(\Diamond)$: where b is a fresh nominal (not applicable if F is a nominal)

$(\Diamond^-)$: where b is a fresh nominal

$$\frac{a{:}\,\mathsf{A}F}{b{:}\,F}\ (\mathsf{A}) \qquad \frac{a{:}\,\mathsf{E}F}{b{:}\,F}\ (\mathsf{E})$$

(A): where b occurs in the branch

(E): where b is a fresh nominal

$$\frac{\begin{array}{c}[\mathcal{B}]\\(n)\, a{:}\,b\end{array}}{\mathcal{B}[b/a]}\ (=)$$

Table 2: Expansion rules: first group

A tableau is expanded by application of the rules in Tables 2 and 3, which are applied to a given branch. Most rules are standard, and their reading is standard too. The *equality rule* (=) does not add any node to the branch, but modifies the labels of its nodes. The schematic formulation of this rule in Table 2 indicates that it can be fired whenever a branch $\mathcal{B}$ contains a *nominal equality* of the form $a\colon b$ (with $a \neq b$); as a result of the application of the rule, every node label F in $\mathcal{B}$ is replaced by $F[b/a]$.

The rules of Table 2 are the same as those presented in [7], but for the fact that the modal rules ($\Box$, $\Box^-$, $\Diamond$ and $\Diamond^-$) are here reformulated to address the multi-modal case.

The rules $\Box$, $\Box^-$ and A are called *universal rules.* When the A rule is applied producing a node labelled by a formula of the form $b : F$, it is said to *focus* on b (and b is the focused nominal of the inference). The $\Diamond$, $\Diamond^-$ and E rules are called *blockable rules*, formulae of the form $a\colon \Diamond_R F$, where F is not a nominal, $a\colon \Diamond_R^- F$, and $a\colon \mathsf{E}F$ are *blockable formulae* and a node labelled by a blockable formula is a *blockable node.*

Sometimes, the rules Link_1 and Link_2 of Table 3 will be generically called the Link rules, and analogously, the Trans_1, Trans_2, Trans_1^- and Trans_2^- rules are called the Trans rules. The four Trans rules of Table 3 deal with transitive relations and can be seen as reformulations (in the presence of inclusion assertions) of the $\Box$ rule for transitive modal logics (particular cases of such rules are when $R = S$). The two Link rules deal with inclusion assertions.

$$\frac{(n)\, a\colon \Diamond_R b \quad (i)\, R \sqsubseteq S}{(m)\, a\colon \Diamond_S b}\ (\mathsf{Link}_1) \qquad \frac{(n)\, a \Diamond_R b \quad (i)\, R^- \sqsubseteq S}{(m)\, b\colon \Diamond_S a}\ (\mathsf{Link}_2)$$

$$\frac{(n)\, a\colon \Box_S F \quad (m)\, a\colon \Diamond_R\, b \quad (t)\, \mathsf{Trans}(R) \quad (i) R \sqsubseteq S}{(k)\, b\colon \Box_R F}\ (\mathsf{Trans}_1)$$

$$\frac{(n)\, a\colon \Box_S^- F \quad (m)\, b\colon \Diamond_R\, a \quad (t) \mathsf{Trans}(R) \quad (i) R \sqsubseteq S}{(k)\, b\colon \Box_R^- F}\ (\mathsf{Trans}_1^-)$$

$$\frac{(n)\, a\colon \Box_S F \quad (m)\, b\colon \Diamond_R\, a \quad (t)\, \mathsf{Trans}(R) \quad (i) R^- \sqsubseteq S}{(k)\, b\colon \Box_R^- F}\ (\mathsf{Trans}_2)$$

$$\frac{(n)\, a\colon \Box_S^- F \quad (m)\, a\colon \Diamond_R\, b \quad (t) \mathsf{Trans}(R) \quad (i) R^- \sqsubseteq S}{(k)\, b\colon \Box_R F}\ (\mathsf{Trans}_2^-)$$

Table 3: Expansion rules: second group

The premiss n of either the $\Box$, $\Box^-$, or one of the Trans rules is called the *major premiss*, and m is the *minor premiss* of the rule. In an application of a Link rule, n is the *logical premiss.* The premisses i and t, in all rules of Table 3, are the *side premisses* of the rules.

The first node of a branch $\mathcal{B}$ is called the *top node* and its label the *top formula* of $\mathcal{B}$. Nominals occurring in the top formula are called *top nominals.* The notion of top nominal is relative to a tableau branch, because applications of the equality rule may change the top formula, hence the set of top nominals.

A branch is *closed* whenever it contains, for some nominal a, either a pair of nodes $(n)\, a\colon p$, $(m)\, a\colon \neg p$ for some $p \in \mathsf{PROP}$, or a node $(n)\, a\colon \neg a$. As usual, we assume that a closed branch is never expanded further on. A branch which is not closed is *open.* A branch is *complete* when it cannot be further expanded.

3.2 Blocking and other restrictions on rule application

Termination is achieved by means of a loop-checking mechanism using nominal renaming. Substantially, in order for a node $(n)\, F$ to (directly) block $(m)\, G$ in a branch $\mathcal{B}$, it must be the case that $G = F[a_1/b_1, \ldots, a_n/b_n]$, where $a_1, \ldots, a_n, b_1 \ldots, b_n$ are *non-top* nominals such that, for all $i = 1, \ldots, n$, a_i and b_i label the same set of propositions in PROP and the same formulae of the form $\Box_R F$ and $\Box_R^- F$.

More precisely:

Definition 1 (Nominal compatibility and mappings)**.** *If $\mathcal{B}$ is a tableau branch, then:*

1. *two nominals a and b are* compatible *in $\mathcal{B}$ if they label the same propositions in* PROP *and the same formulae of the form $\nabla_R F$, for $\nabla \in \{\Box, \Box^-\}$.*

2. *A mapping π for $\mathcal{B}$ is an* injective *function from* non-top *nominals to* non-top *nominals such that for all a, a and $\pi(a)$ are compatible in $\mathcal{B}$.*

3. *Mappings are extended to act on formulas in the obvious way: if π is a mapping and F a formula, $\pi(F)$ is the formula obtained by substituting $\pi(a)$ for a in F, for every nominal a. A mapping π for $\mathcal{B}$ maps a formula F to a formula G if:*

   (a) $\pi(F) = G$;

   (b) π is the identity for all nominals which do not occur in F.

4. *A formula F can be mapped to a formula G in $\mathcal{B}$ if there exists a mapping π for $\mathcal{B}$ mapping F to G.*

The application of the blockable rules is restricted by blocking conditions: a direct blocking condition, which forbids the application of a blockable rule to a node n, whenever the label of a previous node can be mapped to label(n); and also an indirect blocking condition.

Indirect blocking relies on a partial order $\prec_{\mathcal{B}}$, called the *offspring relation*, on the nodes of the branch $\mathcal{B}$, which arranges them into a family of trees, where non-terminal nodes are blockable nodes. Every tree is rooted at a node called a *root node* (a node with no *parents* w.r.t. the offspring relation). When a blockable rule is applied, the generated nodes are *children* (w.r.t. the offspring relation) of the expanded node. All the other rules generate *siblings* of one of the premisses of the inference (two nodes are siblings either if they are both root nodes or they have the same parent).

Properly, the offspring relation and blockings are defined by a mutual recursion on branch construction: if $\mathcal{B}'$ is a branch obtained by expanding $\mathcal{B}$, the definition of $\prec_{\mathcal{B}'}$ assumes that the set of blocked nodes in $\mathcal{B}$ is already defined, and indirectly blocked nodes in $\mathcal{B}$ depend on the relation $\prec_{\mathcal{B}}$. This is due to the presence of the A rule, for which a *minor premiss* must be defined, since nodes added to a branch $\mathcal{B}$ by an application $\mathcal{I}$ of the A rule are siblings of such a minor premiss (in the new branch $\mathcal{B}'$ obtained from the expansion); but, in order to determine the minor premiss of $\mathcal{I}$ it is necessary to know which nodes are blocked in $\mathcal{B}$.

The presentation that follows is somewhat simplified, and the reader is referred to [7] for the more formal approach. Let us assume that when the A rule is applied, beyond the premiss shown in Table 2, the branch contains a node called the minor premiss of the rule application (which will be defined further on, in Definition 5).

Definition 2 (Offspring relation)**.** *Let $\mathcal{B}$ be a tableau branch.*

1. *Every node already contained in the initial branch from which $\mathcal{B}$ is obtained (i.e. its top node and all the nodes labelled by assertions) is a root node.*

2. *If a node n has been added to $\mathcal{B}$ by application of a blockable rule to node m, then $m \prec_{\mathcal{B}} n$ (n is a child of m and m is the parent of n).*

3. *If n has been added to $\mathcal{B}$ by application of either a universal rule or one of the* Trans *rules, whose minor premiss is m, then n is a sibling of m (i.e., if m is a root node, then n is a root node too; otherwise, if $k \prec_{\mathcal{B}} m$, then $k \prec_{\mathcal{B}} n$).*

4. *If n has been added to $\mathcal{B}$ by application of any other rule of table 2 (i.e. any other single-premiss rule) to node m, then n is a sibling of m.*

5. *If n has been added to $\mathcal{B}$ by application of one of the* Link *rules, then n is a sibling of the logical premiss of the inference.*

The notions of direct and indirect blocking can now be defined.

Definition 3 (Direct and indirect blocking)**.** *Let $\mathcal{B}$ be a tableau branch. The set of directly and indirectly blocked nodes in $\mathcal{B}$ is defined by induction on the (total) order $<$ on the nodes of $\mathcal{B}$:*

- *n is blocked if it is either directly or indirectly blocked.*
- *n is directly blocked by m if n is a blockable node, $m < n$, m is not blocked and* label(m) *can be mapped to* label(n) *in $\mathcal{B}$; n is directly blocked in $\mathcal{B}$ if it is directly blocked by some m in $\mathcal{B}$.*
- *n is indirectly blocked if it is not directly blocked and it has an ancestor w.r.t. $\prec_{\mathcal{B}}$ which is blocked.*

An indirectly blocked node is called a phantom node *(or, simply, a phantom).*

The application of the expansion rules is restricted by the conditions defined next. Restrictions **R1**–**R4** are the same as those formulated in [7].[1] The restrictions concerning the new rules are formulated apart (**R5**–**R6**).

Definition 4 (Restrictions on the expansion rules)**.** *The expansion of a tableau branch $\mathcal{B}$ is subject to the following restrictions:*

R1. *no node labelled by a formula already occurring in $\mathcal{B}$ as the label of a* non-phantom *node is ever added to $\mathcal{B}$.*

R2. *A node n labelled by $a\colon\Diamond_R F$ (or $a\colon\Diamond_R^- F$) cannot be expanded if $\mathcal{B}$ contains non-phantom nodes labelled by $a\colon\Diamond_R b$ ($b\colon\Diamond_R a$) and $b\colon F$ for some nominal b.*

A node $(n)\, a\colon \mathsf{E}F$ cannot be expanded in a branch $\mathcal{B}$, if it already contains a non-phantom node labelled by $b\colon F$ for some nominal b.

R3. *A phantom node cannot be expanded by means of a single-premiss rule (including the equality rule), nor can it be used as the minor premiss of a universal rule.*

R4. *A blockable node n cannot be expanded if it is directly blocked in $\mathcal{B}$.*

R5. *A phantom node cannot be used as the minor premiss of any of the* Trans *rules.*

R6. *A phantom node cannot be used as the logical premiss of any of the* Link *rules.*

Note that, as a particular case of restriction **R3**, the A rule cannot focus on a nominal which only occurs in phantom nodes in the branch.

Finally, we only need to define the minor premiss of an application of the A rule.

Definition 5. *If $\mathcal{B}$ is obtained from $\mathcal{B}'$ by means of an application of the* A *rule focusing on the nominal b, then the minor premiss of such an application of the* A *rule is the first non-phantom node in $\mathcal{B}'$ where b occurs.*

Thanks to restriction **R3**, every application of the A rule has a minor premiss.

We conclude with a simple example, showing the interplay between the Trans and Link rules. The notation $n \leadsto^{\mathcal{R}} m$ or $(n_1,\dots,n_k) \leadsto^{\mathcal{R}} m$, used in the rightmost column, is used to mean that the addition of node m is due to the application of rule $\mathcal{R}$ to node n (or nodes $n_1,\dots,n_k$).

Example 1. *Figure 1 shows a closed one-branch tableau for the formula $\Diamond_S\Diamond_S p \wedge \Box_S\neg p$, together with the assertions* Trans(R)*, $R \sqsubseteq S$, $S \sqsubseteq R$. Nodes 0–4 constitute the initial tableau. The branch is closed because of nodes* 11 *and* 15.

(0)	$a\colon\Diamond_S\Diamond_S p \wedge \Box_S\neg p$		(8)	$a\colon\Diamond_S b$	$6 \leadsto^{\Diamond} 8$
(1)	Trans(R)		(9)	$b\colon\Diamond_S p$	$6 \leadsto^{\Diamond} 9$
(2)	$R \sqsubseteq S$		(10)	$b\colon\Diamond_S c$	$9 \leadsto^{\Diamond} 10$
(3)	$S \sqsubseteq R$		(11)	$c\colon p$	$9 \leadsto^{\Diamond} 11$
(4)	$R \sqsubseteq R$	Rel_0	(12)	$a\colon\Diamond_R b$	$(8,3) \leadsto^{\mathsf{Link}_1} 12$
(5)	$S \sqsubseteq S$	Rel_0	(13)	$b\colon\Diamond_R c$	$(10,3) \leadsto^{\mathsf{Link}_1} 13$
(6)	$a\colon\Diamond_S\Diamond_S p$	$0 \leadsto^{\wedge} 6$	(14)	$b\colon\Box_R\neg p$	$(7,12,1,2) \leadsto^{\mathsf{Trans}_1} 14$
(7)	$a\colon\Box_S\neg p$	$0 \leadsto^{\wedge} 7$	(15)	$c\colon\neg p$	$(14,13) \leadsto^{\Box} 15$

Figure 1: Example 1.

[1] The restriction named **R5** in [7] is subsumed by **R3**, and what was called **R6** is now included into **R2**.

4 A variant of the tableau calculus

4.1 A compact description of the expansion rules

The expansion rules of Table 3 are similar to those used to treat transitive roles and role hierarchies in description logics (see, for instance [11, 12]), as well as the rules proposed in [14] to treat the corresponding concepts in the language of hybrid logic. Such a similarity can be more easily recognized if the following notational conventions are adopted.

Definition 6. *Relation symbols will also be called* positive relations *and the inverse of relation symbols* negative relations. *A* relation *is either a positive or negative relation. Relations are denoted by boldface letters:* $\mathbf{R}$ *is a meta-symbol used to denote either R itself or its inverse R^-. Two relations have* the same sign *if they are either both positive or both negative; otherwise they have different signs.*

The notations on the left below are shorthand for the formulae or assertions on the right:

$$a \Rightarrow_{\mathbf{R}} b \quad = \quad \begin{cases} a\colon \Diamond_R b & \text{if } \mathbf{R} = R \\ b\colon \Diamond_R a & \text{if } \mathbf{R} = R^- \end{cases}$$

$$a\colon \Diamond_{\mathbf{R}} F \quad = \quad \begin{cases} a\colon \Diamond_R F & \text{if } \mathbf{R} = R \\ a\colon \Diamond_R^- F & \text{if } \mathbf{R} = R^- \end{cases}$$

$$a\colon \Box_{\mathbf{R}} F \quad = \quad \begin{cases} a\colon \Box_R F & \text{if } \mathbf{R} = R \\ a\colon \Box_R^- F & \text{if } \mathbf{R} = R^- \end{cases}$$

$$\mathbf{R} \sqsubseteq \mathbf{S} \quad = \quad \begin{cases} R \sqsubseteq S & \text{if } \mathbf{R} \text{ and } \mathbf{S} \text{ have the same sign} \\ R^- \sqsubseteq S & \text{if } \mathbf{R} \text{ and } \mathbf{S} \text{ have different signs} \end{cases}$$

$$\mathsf{Trans}(\mathbf{R}) \quad = \quad \mathsf{Trans}(R) \text{ in both cases } \mathbf{R} = R \text{ and } \mathbf{R} = R^-$$

With these notations, the rules Rel_1–Rel_4 of Table 1 can be represented by a single "meta-rule":

$$\frac{\mathbf{R} \sqsubseteq \mathbf{S} \quad \mathbf{S} \sqsubseteq \mathbf{P}}{\mathbf{R} \sqsubseteq \mathbf{P}}$$

The same can be said of the the $\Box$ and $\Box^-$ rules:[2]

$$\frac{(n) a\colon \Box_{\mathbf{R}} F \quad (m)\, a \Rightarrow_{\mathbf{R}} b}{(k)\, b\colon F}$$

and the rules of Table 3 can be given a similar more compact presentation, as shown in Table 4. Note that, in the Link rule, R is a relation symbol (never a converse), and that $R \sqsubseteq S^-$ is a shorthand for $R^- \sqsubseteq S$, therefore, if $\mathbf{S} = S^-$, then $R \sqsubseteq \mathbf{S} = R^- \sqsubseteq S$. The instance of the Trans rule where $\mathbf{S}$ and $\mathbf{R}$ are both positive is the Trans_1 rule; if $\mathbf{S}$ and $\mathbf{R}$ are both negative, it is the Trans_1^- rule; the Trans_2 and Trans_2^- rules are the cases where $\mathbf{S} = S$ and $\mathbf{R} = R^-$, and $\mathbf{S} = S^-$ and $\mathbf{R} = R$, respectively.

$$\frac{(n)\, a\colon \Diamond_R b \quad (i)\, R \sqsubseteq \mathbf{S}}{(m)\, a \Rightarrow_{\mathbf{S}} b}\ (\mathsf{Link})$$

$$\frac{(n)\, a\colon \Box_{\mathbf{S}} F \quad (m)\, a \Rightarrow_{\mathbf{R}} b \quad (t)\, \mathsf{Trans}(\mathbf{R}) \quad (i) \mathbf{R} \sqsubseteq \mathbf{S}}{(k)\, b\colon \Box_{\mathbf{R}} F}\ (\mathsf{Trans})$$

Table 4: A compact presentation of the rules of Table 3.

This way of presenting the expansion rules highlights the closeness of the Trans rule to the corresponding rule used in description logics, where in fact "roles" include both *role names* (corresponding

[2] The $\Diamond$ and $\Diamond^-$ rules cannot be represented in an analogous way, because of the additional restriction in the $\Diamond$ rule.

to relation symbols) and the inverse of role names, and inverse roles may also occur in role inclusion axioms.[3]

The abbreviation $a \Rightarrow_{\mathbf{R}} b$, however, does not have exactly the same meaning as the corresponding premiss used in the rule treating transitivity in description logics [11, 12] (a similar approach is adopted in [14]), consisting of the meta-notion "b is an R-neighbour of a".[4] In the setting of the present work, "b is an $\mathbf{R}$-neighbour of a" would amount to saying that there exists a (positive or negative) relation $\mathbf{S}$ such that $\mathbf{S} \sqsubseteq \mathbf{R}$ and $a \Rightarrow_{\mathbf{S}} b$.

There are two main differences between the two approaches. First of all, the semantical notion of accessibility between two states is here given a "canonical representation" in the object language (a choice already made in [6, 7]): the fact that a state a is R-related to b is represented by the *relational formula* $a{:}\Diamond_R b$. Though semantically equivalent to $b{:}\Diamond^-_R a$, the latter is not a relational formula, i.e. it is not the canonical representation of an R-relation. This is reflected by the fact that the $\Diamond$ rule cannot be applied to a relational formula, while $b{:}\Diamond^-_R a$ can be expanded by means of the $\Diamond^-$ rule. Moreover, in the present work, the notation $a \Rightarrow_{\mathbf{R}} b$ is only an abbreviation for a relational formula, which does not take subrelations into account: it may be the case that $a \Rightarrow_{\mathbf{S}} b$ for some $\mathbf{S} \sqsubseteq \mathbf{R}$, and yet $a \Rightarrow_{\mathbf{R}} b$ does not hold.

The fact that, in the present work, no meta-notion is used to represent "R-neighbours" is responsible for the presence of the Link rules, that have no counterpart in [11, 12, 14].

4.2 Getting rid of the Link rules

Actually, new relational nodes derivable by means of the Link rules are only needed when they can be premisses of either the $\Box$, $\Box^-$ or the Trans rules. They can be dispensed with and implicitly embodied in such rules. The tableau expansion rules presented below are actually true reformulations of the corresponding rules in tableaux for description logics. The rules of Table 5 replace the $\Box$ and $\Box^-$ rules of Table 2 and all the rules of Table 3. They are presented by use of the abbreviations introduced in Definition 6, and such abbreviations will be used from now on, even with an abuse of terminology, speaking of "the $\Box_i$ rule" to mean one of the four rules which might be called $\Box_1, \ldots, \Box_4$, and "the Trans_i rule" stands for one of the eight ones $\mathsf{Trans}_1, \ldots, \mathsf{Trans}_8$.

$$\frac{(n)\, a{:}\Box_{\mathbf{R}} G \quad (m)\, a \Rightarrow_{\mathbf{S}} b \quad (i)\, \mathbf{S} \sqsubseteq \mathbf{R}}{(k)\, b{:}G}\ (\Box_i)$$

$$\frac{(n)\, a{:}\Box_{\mathbf{S}} F \quad (m)\, a \Rightarrow_{\mathbf{P}} b \quad (i')\mathbf{P} \sqsubseteq \mathbf{R} \quad (t)\, \mathsf{Trans}(\mathbf{R}) \quad (i)\mathbf{R} \sqsubseteq \mathbf{S}}{(k)\, b{:}\Box_{\mathbf{R}} F}\ (\mathsf{Trans}_i)$$

Table 5: Variant of the expansion rules

The $\Box_i$ rules are universal rules. The premiss n of either a $\Box_i$ or a Trans_i rule is its major premiss, the premiss m the minor premiss and the others are the side premisses of the inference.

The offspring relation is defined like in Definition 2, but for the last item, which is obviously omitted. Being universal rules, the $\Box_i$ rules are taken into account by item 3; in the same item, the Trans rules have to be replaced by the Trans_i rules. The same happens in Definition 4: **R3** now affects also the $\Box_i$ rules, **R5** concerns the Trans_i rules and **R6** is omitted.

Example 2. *Figure 2 shows a closed one-branch tableau obtained with the variant of the expansion rules for the same formula and assertions of Example 1.*

The next results establishes the straightforward relation existing between the calculus presented in Section 3 and its variant.

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[3]The modal counterpart of such an approach would use modal operators of the form $\Box_{R^-}$ and $\Diamond_{R^-}$ instead of $\Box^-_R$ and $\Diamond^-_R$.

[4]The definition of R-neighbour (where R is a "role", i.e. either a role name or the converse of a role name) is the following: y is an R-neighbour of x if there exists a *role* $S \sqsubseteq R$ such that one of the following cases holds: (1) there is an arc from x to y labelled by S; (2) there is an arc from y to x labelled by the inverse of S. Semantically, y is an R-neighbour of x if x is R-related to y, whichever representation is given for this fact, but however taking into account the role hierarchy.

(0)	$a\colon \Diamond_S\Diamond_S p \wedge \Box_S\neg p$		(7)	$a\colon \Box_S\neg p$	$0 \leadsto^{\wedge} 7$
(1)	$\mathsf{Trans}(R)$		(8)	$a\colon \Diamond_S b$	$6 \leadsto^{\Diamond} 8$
(2)	$R \sqsubseteq S$		(9)	$b\colon \Diamond_S p$	$6 \leadsto^{\Diamond} 9$
(3)	$S \sqsubseteq R$		(10)	$b\colon \Diamond_S c$	$9 \leadsto^{\Diamond} 10$
(4)	$R \sqsubseteq R$	Rel_0	(11)	$c\colon p$	$9 \leadsto^{\Diamond} 11$
(5)	$S \sqsubseteq S$	Rel_0	(12)	$b\colon \Box_R\neg p$	$(7, 8, 3, 1, 2) \leadsto^{\mathsf{Trans}_i} 12$
(6)	$a\colon \Diamond_S\Diamond_S p$	$0 \leadsto^{\wedge} 6$	(13)	$c\colon \neg p$	$(12, 10, 3) \leadsto^{\Box_i} 14$

Figure 2: Example 2.

Lemma A. *The rules of Table 5 are all derivable from the rules of Tables 2 and 3, preserving the offspring relation.*

Proof. The $\Box_i$ rule can be derived by means of the Link and $\Box$ (or $\Box^-$) rule, and the Trans_i rules by use of the suitable Link and the Trans rules, as shown in Figure 3. Furthermore, if the minor premiss m of the $\Box_i$ or Trans_i rule is a phantom, then the Link rule cannot be applied either. If m is not a phantom, the conclusion m' of the Link rule is a sibling of m, therefore it is not a phantom either, and the $\Box$ ($\Box^-$) or Trans rules can be applied.

Finally, the conclusion k of both derivations above is a sibling of m, as required for the $\Box_i$ and Trans_i rules. $\Box$

$$\dfrac{(n)\ a\colon \Box_{\mathbf{R}} G \qquad \dfrac{(m)\ a \Rightarrow_{\mathbf{S}} b \quad (i)\ \mathbf{S} \sqsubseteq \mathbf{R}}{(m')\ a \Rightarrow_{\mathbf{R}} b}\,(\mathsf{Link})}{(k)\ b\colon G}\,(\Box^{[-]})$$

$$\dfrac{(n)\ a\colon \Box_{\mathbf{S}} F \qquad \dfrac{(m)\ a \Rightarrow_{\mathbf{P}} b \quad (i')\mathbf{P} \sqsubseteq \mathbf{R}}{(m')\ a \Rightarrow_{\mathbf{R}} b}\,(\mathsf{Link}) \qquad (t)\ \mathsf{Trans}(\mathbf{R}) \quad (i)\mathbf{R} \sqsubseteq \mathbf{S}}{(k)\ b\colon \Box_{\mathbf{R}} F}\,(\mathsf{Trans})$$

Figure 3: Derivation of the rules of Table 5 from those of Tables 2 and 3

5 Termination and completeness with transitive relations and relation hierarchies

Both calculi presented in this work are trivially sound. Moreover, each of them is complete and terminating, *provided that the initial formula is in the fragment* $\mathit{HL}_m(@, \downarrow, \mathit{E}, \Diamond^-) \setminus \downarrow\Box$. This section contains a succinct outline of the termination and completeness proofs. The whole proofs are quite long already for the calculus defined in [7], so they are just summarized below and then the integrations and modifications needed to add assertions are shown. In order to make the presentation as readable as possible, however, statements and definitions are fully reported, when needed to understand the changes w.r.t. the proofs given in [7]. The numbering of lemmas will be the same as in [7], so that the reader can easily find them for comparison, and new intermediate results and definitions are numbered autonomously.

By Lemma A, if the calculus presented in Section 3 terminates, then also its variant does; and if the variant of Section 4 is complete, then so is the calculus of Section 3. Section 5.1 is devoted to show termination of the first calculus, and Section 5.2 shows that the variant of the calculus is complete. In order to make proofs more compact, they will make use of the notations introduced in Section 4.1, when possible.

In what follows, it is always assumed that the initial formula of the tableau is in the fragment $\mathsf{HL}_m(@, \downarrow, \mathsf{E}, \Diamond^-) \setminus \downarrow\Box$, even when it is not explicitly stated.

The key result used to prove both termination and completeness is a form of subformula property. In the presence of subroles, the definition of the set of subformulae of a given formula F has to be widened, including among them all the formulae of the form $\Box_R G$ and $\Box^-_R G$, for every subformula $\Box_S G$ or $\Box^-_S G$ of F and for every relation symbol R in the language.

Definition D1. *If F is a formula, then G is a subformula of F if and only if either $F = G$ or one of the following conditions holds:*

- $F = F_1 \star F_2$, *for* $\star \in \{\wedge, \vee\}$ *and G is a subformula of* F_i*;*
- *either* $F = t\colon F_0$ *or* $F = \downarrow x.F_0$ *or* $F = \nabla F_0$ *for* $\nabla \in \{\mathsf{A}, \mathsf{E}, \diamond_R, \diamond_R^-\}$, *and G is a subformula of* F_0*;*
- $F = \Box_{\mathbf{R}} F_0$, *for some* relation $\mathbf{R}$, *and G is a either a subformula of F_0 or* $G = \Box_{\mathbf{S}} F_0$ *for some* relation $\mathbf{S}$ *in the the language.*

If $\mathcal{B}$ is a tableau branch and $a_0\colon F_0$ its top formula, $\mathrm{Subf}(\mathcal{B})$ *is the set of the subformulae of F_0, and*

$$\mathrm{Cmp}(\mathcal{B}) = (\mathrm{Subf}(\mathcal{B}) \cap \mathsf{PROP}) \cup \{\Box_{\mathbf{R}} G \mid \Box_{\mathbf{R}} G \in \mathrm{Subf}(\mathcal{B})\}$$

The following result bounds the number of subformulae of a given formula.

Lemma B. *Let F be a formula in a language with M relation symbols, and $|F| = N$ the size of F. Then F has no more than $2 \times M \times N$ subformulae.*

Proof. The number of "standard subformulae" of F is bounded by N. Each of them, if it has either form $\Box_R G$ or $\Box_R^- G$, has $2 \times M$ more subformulae (as defined in Definition D1). Therefore, F has no more than $2 \times M \times N$ subformulae. $\Box$

With this modification, the main property of the calculus, stated by Lemma 4 in [7] still holds for both variants of the calculus. It uses the notion of *instance* of a formula F, that is any expression obtained by uniformly replacing every free variable of F with some nominal.

Lemma 4 (Subformula properties)**.** *For any formula $a\colon F$ occurring in a branch $\mathcal{B}$ which is not a relational formula, F is an instance of a formula in* $\mathrm{Subf}(\mathcal{B})$*.*

Moreover, assuming that the initial formula of the branch is in the fragment $\mathsf{HL}\backslash\downarrow\Box$*, if F is a universal formula, then* $F \in \mathrm{Subf}(\mathcal{B})$*.*

Proof. The proof is an induction on the construction of $\mathcal{B}$, which simultaneously proves the following strongest versions of the two properties: if $(n)\, a\colon F$ is a node in $\mathcal{B}$ and $a\colon F$ is not a relational formula, then for any subformula F' of F:

(α) F' is an instance of a formula in $\mathrm{Subf}(\mathcal{B})$, and

(β) if F' is a universal formula, then $F' \in \mathrm{Subf}(\mathcal{B})$.

The induction step of the corresponding proof in [7] can easily be extended with the cases where the branch $\mathcal{B}$ is obtained from $\mathcal{B}'$ by application of one of the new rules. We show below the treatment of all new rules of Tables 3 and 5.

1. $\mathcal{B}$ is obtained by application of one of the Link rules. The newly added node is labelled by a relational formula, so α and β are vacuously true.
2. If $\mathcal{B}$ is obtained by application of one of the Trans rules of Table 3 or the Trans_i rule of Table 5, then α and β directly follow from the induction hypothesis.
3. The $\Box_i$ rules of Table 5 are treated exactly like the cases of the $\Box$ and $\Box^-$ rules in [7], the minor and side premisses being irrelevant. $\Box$

5.1 Termination

Termination of the calculus presented in Section 3 is proved, like in [7], by showing that the nodes of a branch $\mathcal{B}$ are arranged by the offspring relation into a bounded sized set of trees, each of which has bounded width and bounded depth. Hence any tableau branch $\mathcal{B}$ has a number of nodes that is bounded by a function of the size of the initial formula.

In order to show that, in the forest of trees induced by the offspring relation on the nodes of a branch $\mathcal{B}$, any node has a bounded number of siblings, the key result is is Lemma 5 below. It uses the relation $m \rhd n$, holding between two nodes m and n whenever they are siblings w.r.t. the offspring relation and n has been added to the branch by application of an expansion rule to premisses including m. I.e. $m \rhd n$ if one of the following conditions hold:

- n is added to the branch by application of some non-blockable single premiss rule to m;
- n is added to the branch by application of either a universal rule or a Trans rule whose minor premiss is m;
- n is added to the branch by application of a Link rule whose logical premiss is m.

The relation $\rhd^*$ is the reflexive and transitive closure of $\rhd$. If $n \rhd^* m$, we say that n *produces* m.

The proof of Lemma 5 uses the notions defined as follows. Let M be a set of nominals, F a formula (possibly containing free variables) and Δ a set of formulae.

1. $\mathrm{Clo}(\Delta)$ (the *closure* of Δ) is the set containing all the subformulae of every formula in Δ.
2. An M-instance of F is a ground formula that can be obtained from F by replacing its free variables with elements of M.
3. The set Δ^M is the set containing all the M-instances of every element of Δ.

Note that, though the above definitions are formally the same as in [7], the set denoted by $\mathrm{Clo}(\Delta)$ is larger, because of the new notion of subformula.

Lemma 5. *Let n be a node in a branch $\mathcal{B}$ of a tableau for a formula F, and let $N = |F|$. Then the cardinality of $\Sigma(n) = \{m \mid n \rhd^* m\}$ is bounded by an exponential function $E_w(N)$.*

Proof. The thesis is proved by showing that the label of any node in $\Sigma(n)$ has a *matrix* taken from a bounded stock of formulae, that is built in the language of the branch *at the time n is added to it.* Node labels with the same matrix are always equal, at any construction stage of the branch, so that the cardinality of $\Sigma(n)$ is bounded by the number of such possible matrices, since siblings always have the same phantom/non-phantom status.

Any branch $\mathcal{B}$ in a tableau is the last element of a sequence of branches, where the first one is the initial tableau, and each of the others is obtained from the previous one by application of an expansion rule. Such a sequence will be called the *sequence of branches leading to $\mathcal{B}$.*

Let n be any fixed node in a tableau branch $\mathcal{B}$. The following notations will be used:

1. $\mathcal{B}_1$ is the first branch where n occurs, in the sequence of branches leading to $\mathcal{B}$.
2. $\mathrm{label}_{\mathcal{B}_i}(k)$ is the label of the node k in the branch $\mathcal{B}_i$. This allows one to refer to node labels in different branches.
3. For $1 \le i \le p$, σ_i is the composition of the substitutions applied in the sequence $\mathcal{B}_1, \ldots, \mathcal{B}_i$, by means of the equality rule. Consequently, for each $i > 0$, $\mathrm{label}_{\mathcal{B}_i}(n) = \sigma_i(\mathrm{label}_{\mathcal{B}_1}(n))$.
4. M_n is the set containing all the nominals occurring in $\mathrm{label}_{\mathcal{B}_1}(n)$ and all the top nominals in $\mathcal{B}_1$.
5. Γ_n, Δ_n and S_n are the sets of formulae defined as follows:

$$\begin{array}{rcl} \Gamma_n &=& \{F \mid F \text{ is a universal subformula of the top formula of } \mathcal{B}_1\} \\ \Delta_n &=& \{\mathrm{label}_{\mathcal{B}_1}(n)\} \cup \Gamma_n \\ S_n &=& (\mathrm{Clo}(\Delta_n))^{M_n} \end{array}$$

   i.e. S_n contains all the M_n-instances of every formula in the closure of Δ_n.

6. $\mathcal{F}_n$ is the set defined as follows:

$$\begin{array}{rcl} \mathcal{F}_n &=& \{a{:}\, F \mid a \in M_n \text{ and } F \in S_n\} \cup \\ && \{a{:}\, \Diamond_R b \mid a, b \in M_n \text{ and } R \text{ is any relation symbol in the language}\} \end{array}$$

   Any element of $\mathcal{F}_n$ will be called a *matrix* (this definition is a straightforward extension to the multi-modal case of the corresponding definition in [7]).

The bound $E_w(N)$ on the cardinality of $\Sigma(n)$, computed in [7], is equal to $|\mathcal{F}_n|$. Such a value, in turn, is shown to be equal to $N_0 + N_1^2 \times N^{N+1}$, where N_0 is the maximal number of relational formulae which can be built out of N nominals, and N_1 is the maximal number of subformulae of a formula of size N. In the uni-modal case, $N_0 = N^2$ and $N_1 = N$, while in the multi-modal case $N_0 = N^3$ and $N_1 = 2 \times N^2$ (by Lemma B, since the number of relation symbols in the language is also bounded by N). The computation of the exponential factor is independent of the number of modalities in the language. Therefore, the bound $E_w(N)$ is exponential also in the multi-modal case.

Let m be any node in $\Sigma(n)$, i.e. $n \rhd^* m$. An element F of $\mathcal{F}_n$ is called a *matrix* of m in $\mathcal{B}_i$ if $\text{label}_{\mathcal{B}_i}(m) = \sigma_i(F)$; and F is a matrix of m if it is a matrix of m in all $\mathcal{B}_i$ where m occurs, for $i = 1, \dots, p$. If two nodes m_1 and m_2 have the same matrix, then obviously for all $i = 1, \dots, p$ such that both m_1 and m_2 are in $\mathcal{B}_i$, $\text{label}_{\mathcal{B}_i}(m_1) = \text{label}_{\mathcal{B}_i}(m_2)$.

The proof that the cardinality of $\Sigma(n)$ is bounded by $E_w(N)$, where $E_w(N)$ is the cardinality of $\mathcal{F}_n$, is based on the fact that every node in $\Sigma(n)$ has a matrix:

(α) the label of any node in $\Sigma(n)$ has a matrix in $\mathcal{F}_n$. I.e. if $m \in \Sigma(n)$, then there exists $F \in \mathcal{F}_n$ such that for all $i \geq 1$, if $m \in \mathcal{B}_i$ then $\text{label}_{\mathcal{B}_i}(m) = \sigma_i(F)$.

The proof is by induction on i. We show next the cases of the induction step corresponding to the new rules of Table 3.

(Link) Let $n \rhd^* k$ and m be obtained by an application of one of the Link rules to nodes k and t, with

$$\begin{array}{l}\text{label}_{\mathcal{B}_{i-1}}(k) = \text{label}_{\mathcal{B}_i}(k) = a{:}\, \Diamond_R b \\ \text{label}_{\mathcal{B}_{i-1}}(m) = \text{label}_{\mathcal{B}_i}(m) = a \Rightarrow_{\mathbf{S}} b.\end{array}$$

By the induction hypothesis, $a{:}\, \Diamond_R b = \sigma_i(c{:}\, \Diamond_R d)$ for some $c{:}\, \Diamond_R d \in \mathcal{F}_n$, i.e., $a = \sigma_i(c)$ and $b = \sigma_i(d)$ for $c, d \in M_n$. Since c and d belong to M_n, $c \Rightarrow_{\mathbf{S}} d \in \mathcal{F}_n$. Therefore $c \Rightarrow_{\mathbf{S}} d \in \mathcal{F}_n$ is a matrix of $a \Rightarrow_{\mathbf{S}} b \in \mathcal{F}_n$ in $\mathcal{B}_i$, because $a \Rightarrow_{\mathbf{S}} b = \sigma_i(c \Rightarrow_{\mathbf{S}} d)$.

(Trans) Let $n \rhd^* k$ and m be obtained by an application of one of the Trans rule to nodes k, k', t and i, with

$$\begin{array}{l}\text{label}_{\mathcal{B}_{i-1}}(k') = \text{label}_{\mathcal{B}_i}(k') = a \Rightarrow_{\mathbf{R}} b, \\ \text{label}_{\mathcal{B}_{i-1}}(m) = \text{label}_{\mathcal{B}_i}(m) = b{:}\, \Box_{\mathbf{R}} G.\end{array}$$

By Lemma 4, $\Box_{\mathbf{R}} G \in S_n$. By the induction hypothesis, $a \Rightarrow_{\mathbf{R}} b = \sigma_i(c \Rightarrow_{\mathbf{R}} d)$ for some $c \Rightarrow_{\mathbf{R}} d \in \mathcal{F}_n$, i.e. $b = \sigma_i(d)$ for some $d \in M_n$. Therefore $d{:}\, \Box_{\mathbf{R}} G \in \mathcal{F}_n$ and, since $b{:}\, \Box_{\mathbf{R}} G = \sigma_i(d){:}\, \Box_{\mathbf{R}} G = \sigma_i(d{:}\, \Box_{\mathbf{R}} G)$, $d{:}\, \Box_{\mathbf{R}} G$ is a matrix of m in $\mathcal{B}_i$.

The fact that the cardinality of $\Sigma(n)$ is bounded by $E_w(N)$, where $E_w(N)$ is the cardinality of $\mathcal{F}_n$ is finally proved like in [7], obviously taking now into account also restrictions **R5** and **R6**: Let us assume, by *reductio ad absurdum*, that $\Sigma(n)$ has more than $E_w(N)$ elements. Then, by α, there are at least two distinct elements m_1 and m_2 in $\Sigma(n)$ which have the same matrix F. We may assume *w.l.g.* that $n \leq m_1 < m_2$. Let $\mathcal{B}_k$ be the first branch where m_2 occurs. Since $n < m_2$, there is a node $k \in \Sigma(n)$ such that $n \rhd^* k \rhd m_2$. Given that k produces a node, it is not the major premiss of a universal rule or one of the Trans rules. Moreover, k is not a phantom in $\mathcal{B}_{k-1}$, otherwise one of restrictions **R3**, **R5** or **R6** would be violated. Consequently, m_1 is not a phantom in $\mathcal{B}_{k-1}$ either. But $\text{label}_{\mathcal{B}_k}(m_2) = \sigma_k(F) = \sigma_{k-1}(F) = \text{label}_{\mathcal{B}_{k-1}}(m_1)$ ($\sigma_k = \sigma_{k-1}$ because, clearly, $\mathcal{B}_{k-1}$ has not been expanded by means of the equality rule, which does not add new nodes to the branch). Therefore, the addition of m_2 to $\mathcal{B}_{k-1}$ violates restriction **R1**. $\Box$

Lemma 5 allows for establishing that the number of trees in the forest induced by the offspring relation on the nodes of a tableau branch is bounded by an exponential function of the size of the initial formula, and so is the width of each of such trees. Obviously, now the trees include the single-node ones constituted by nodes labelled by assertions (assertions do not produce any node), but their number is polynomial in the length of the initial formula.

In [7], it is then shown that tree depth is also bounded by an exponential function $E_d(N)$ of N, where N is the size of the initial formula. The value of such a function is computed as $N_1 \times E'_d(N)$, where N_1 is the maximal cardinality of the set of subformulae of the initial formula and $E'_d(N)$ is an exponential function of N. In the mono-modal case, $N_1 = N$, while in the multi-modal one, $N_1 = 2 \times N^2$.

Consequently, both tree width and tree depth increase only of a polynomial factor w.r.t. the uni-modal case.

The rest of the termination proof is independent of the presence of the new expansion rules and multi-modalities, therefore, modulo the replacement of the exact values of $E_w(N)$ and $E_d(N)$, it stays the same and the overall result does not change.

It is worth pointing out that the worst-case complexity of the calculus presented in this work has the same order of magnitude of the calculus in [7]: the nodes of a tableau branch are arranged by $\prec_{\mathcal{B}}$ in a forest of trees, whose number is bounded by an exponential function of the size N of the input formula. Both tree width and tree depth are bounded by exponential functions of N, therefore the number of nodes in a single branch is bounded by a doubly exponential function. Since the cost of blockings is in the order of the branch size, the tableau calculus presented in this work shows that the satisfiability problem for $\mathsf{HL}_m(@, \downarrow, \mathsf{E}, \Diamond^-, \mathsf{Trans}, \sqsubseteq) \setminus \Box\downarrow\Box$ is in 2-NExpTime. It is reasonable to hypothesize that the problem complexity is actually lower,[5] and consequently that, like many other tableau based algorithms, the decision procedure defined in this paper is not worst-case optimal.

5.2 Completeness

In order to prove that the calculus presented in Section 4 is complete, it is shown – like in [7] – how to extend a subset $\mathcal{N}_0$ of any complete and open branch $\mathcal{B}$ in such a way that every directly blocked node is added a suitable "witness". The witness(es) of a blockable node n can be viewed simply as node(s) which could by obtained by application of the corresponding blockable rule to n.[6] The label of each newly added node is obtained from a node in $\mathcal{N}_0$ by suitably renaming non-top nominals. A model of the initial formula can then be extracted from such an extension.

The construction of the extension of $\mathcal{N}_0$ is summarized below. The set $\mathcal{N}_0$ is the union of the non-phantom nodes in $\mathcal{B}$ and the nodes of the form $(n)\, a{:}\, F$, with a occurring in some non-phantom node in $\mathcal{B}$ and $F \in \mathsf{PROP}$ or of the form $\Box_{\mathbf{R}} G$.

The extended set $\mathcal{N}_{\mathcal{B}}^{\infty}$ is built by stages, as the union of a (possibly infinite) sequence of finite extensions of $\mathcal{N}_0$: $\mathcal{N}_0 \subseteq \mathcal{N}_1 \subseteq \mathcal{N}_2 \ldots$. Each set $\mathcal{N}_{i+1}$ is obtained from $\mathcal{N}_i$ by choosing a "blocked" node n in $\mathcal{N}_i$ (with no witnesses). Its label is the renaming of the label of its blocking node m, which always belongs to $\mathcal{N}_0$ and is not blocked. Therefore m has been expanded in $\mathcal{B}$, generating node(s) with a fresh nominal b. Let π_i be the mapping which maps m to n and b_i a new nominal. Then a "nominal renaming" θ_i is defined, extending π_i with $b \mapsto b_i$, and $\mathcal{N}_{i+1}$ extends $\mathcal{N}_i$ by addition of new nodes, each of which is obtained from a node $k \in \mathcal{N}_0$ by application θ_i to its label.

Possibly, new nodes with no witnesses are added, but each of them is blocked by a (non blocked) node in $\mathcal{N}_0$. All the "blocked" nodes in $\mathcal{N}_i$ are stored in *the blocking relation for* $\mathcal{N}_i$, $\mathbf{B}_i$, containing triples of the form (n, m, π), where n is the blocked node (a blockable node without witnesses in $\mathcal{N}_i$), $m \in \mathcal{N}_0$ is not blocked, and π a mapping such that $\pi(\text{label}(m)) = \text{label}(n)$.

Since the strategy to choose the nodes to "unblock" is fair, the set $\mathcal{N}_{\mathcal{B}}^{\infty}$ is such that every blockable node has its witness(es).

The construction enjoys the following properties, which can be proved like in [7] (2 is stated as Lemma 10 in [7], and 3–5 constitute Lemma 11 in [7]):

P1. For all i, the renaming θ_i is an injective function, hence its inverse θ_i^- is defined.

P2. If a nominal b occurs in $\mathcal{N}_0$, then it occurs in some non-phantom node in $\mathcal{B}$.

P3. If $i > 0$ and d is a nominal occurring in $\mathcal{N}_{i-1}$, then no new node added at stage i has a label of the form $d{:}\, p$ for $p \in \mathsf{PROP}$, or $d{:}\, \Box_{\mathbf{R}} G$. As a consequence, if two nominals occurring in $\mathcal{N}_{i-1}$ are compatible in $\mathcal{N}_{i-1}$, then, for any $i > 0$, they stay compatible in $\mathcal{N}_i$ (and in $\mathcal{N}_{\mathcal{B}}^{\infty}$).

P4. If $i > 0$ and θ_i is the mapping used to extend $\mathcal{N}_{i-1}$ to $\mathcal{N}_i$, then for every nominal d occurring in $\mathcal{N}_i$, d and $\theta_i(d)$ are compatible in $\mathcal{N}_i$.

P5. For every triple $(n, m, \pi) \in \mathbf{B}_i$ (i.e. the node $n \in \mathcal{N}_i$ is "blocked" by $m \in \mathcal{N}_0$ by means of the mapping π) and for every nominal d occurring in $\mathcal{N}_i$, d and $\pi(d)$ are compatible in $\mathcal{N}_i$.

[5] The satisfiability problem for $\mathsf{HL}(@, \downarrow, \mathsf{E}, \Diamond^-) \setminus \downarrow\Box$ is in 2ExpTime [20], and the complexity of the concept satisfiability problem in description logics does not increase with the addition of transitive roles and role hierarchies.

[6] Actually, in [7], witnesses are nominals and not nodes, but this detail can be ignored here.

In order to build a model of $\mathcal{N}_{\mathcal{B}}^{\infty}$, such a set is shown to enjoy a form of *saturation property* for non-phantom nodes: it is consistent (there are no labels of the form $a{:}\neg a$, or both $a{:}p$ and $a{:}\neg p$), it does not contain non-trivial equalities, and, for any node or pair of nodes in $\mathcal{N}_0$ that could be the premiss(es) of some expansion rule other than blockable ones, its expansion(s) are also in $\mathcal{N}_0$. Such a notion is defined below. The definition is the same as in [7], but for the reformulation of item 8 to the multi-modal case and the addition of the last items 11–14.

We abuse notation, writing $F_1, F_2, \dots \in \mathcal{N}_i$, meaning that there exist nodes in $\mathcal{N}_i$ labelled by $F_1, F_2, \dots$, respectively

Definition 14. *Let $\mathcal{B}$ be a complete and open branch and $\mathcal{N}_i$ an element of the sequence leading to the construction of $\mathcal{N}_{\mathcal{B}}^{\infty}$. The set $\mathcal{N}_i$ is* pseudo-saturated *with respect to $\mathbf{B}_i$ if it satisfies the following properties:*

1. *no node in $\mathcal{N}_i$ is labelled by a formula of the form $a{:}\neg a$;*
2. *there are no pairs of nodes labelled by formulae of the form $a{:}p$ and $a{:}\neg p$, for $p \in \mathsf{PROP}$;*
3. *if any node in $\mathcal{N}_i$ is labelled by a formula of the form $a{:}d$ (where a and d are nominals), then $a = d$;*
4. *if $a{:}F \wedge G \in \mathcal{N}_i$ then, $a{:}F \in \mathcal{N}_i$ and $a{:}G \in \mathcal{N}_i$;*
5. *if $a{:}F \vee G \in \mathcal{N}_i$, then either $a{:}F \in \mathcal{N}_i$ or $a{:}G \in \mathcal{N}_i$;*
6. *if $a{:}d{:}F \in \mathcal{N}_i$, then $d{:}F \in \mathcal{N}_i$;*
7. *if $a{:}\downarrow x.F \in \mathcal{N}_i$, then $a{:}F[a/x] \in \mathcal{N}_i$;*
8. *if $a{:}\Diamond_{\mathbf{R}} F \in \mathcal{N}_i$, F is not a nominal, and $\mathbf{B}_i$ contains no triple of the form (n, n', π) (i.e. n is not blocked in $\mathbf{B}_i$), then $a \Rightarrow_{\mathbf{R}} d,\ d{:}F \in \mathcal{N}_i$, for some nominal d (i.e. n has a witness in $\mathcal{N}_i$);*
9. *if $a{:}\mathsf{E}F \in \mathcal{N}_i$ and $\mathbf{B}_i$ contains no triple of the form (n, n', π), then $d{:}F \in \mathcal{N}_i$ for some nominal d (i.e. n has a witness in $\mathcal{N}_i$);*
10. *if $a{:}\mathsf{A}F \in \mathcal{N}_i$ and d occurs in $\mathcal{N}_i$, then $d{:}F \in \mathcal{N}_i$;*
11. *$R \sqsubseteq R \in \mathcal{N}_i$ for any $R \in \mathsf{REL}$.*
12. *If $\mathbf{R} \sqsubseteq \mathbf{S},\ \mathbf{S} \sqsubseteq \mathbf{P} \in \mathcal{N}_i$, then $\mathbf{R} \sqsubseteq \mathbf{P} \in \mathcal{N}_i$.*
13. *If $a{:}\Box_{\mathbf{R}} F,\ a \Rightarrow_{\mathbf{S}} d,\ \mathbf{S} \sqsubseteq \mathbf{R} \in \mathcal{N}_i$, then $d{:}F \in \mathcal{N}_i$.*
14. *if $a{:}\Box_{\mathbf{S}} F,\ a \Rightarrow_{\mathbf{R}} b,\ \mathbf{P} \sqsubseteq \mathbf{R},\ \mathsf{Trans}(\mathbf{R}),\ \mathbf{R} \sqsubseteq \mathbf{S} \in \mathcal{N}_i$, then $b{:}\Box_{\mathbf{R}} F \in \mathcal{N}_i$;*

Lemma 12. *Let $\mathcal{B}$ be a complete and open branch and $\mathcal{N}_i$ an element of $\mathcal{S}_{\mathcal{B}}$. Then $\mathcal{N}_i$ is pseudo-saturated with respect to $\mathbf{B}_i$.*

Proof. First of all, we observe that clauses 11 and 12 hold because, since $\mathcal{B}$ is complete, all rules of Table 1 have been applied as far as possible when building the initial tableau. They generate top (hence non-phantom) nodes, which belong to $\mathcal{N}_0 \subseteq \mathcal{N}_i$ for all i.

For the other clauses, the proof is by induction on i. Both the base case and the induction step of the corresponding proof in [7] (possibly reformulated for the multi-modal case) must be completed with the new cases: 13 and 14. Their treatment is actually very similar to the cases dealing with the $\Box$ and $\Box^-$ rules in [7], in both the base case and the induction step.

16. *Base.* If $(m)\, a \Rightarrow_{\mathbf{S}} d \in \mathcal{N}_0$, then m is not a phantom in $\mathcal{B}$. If also $a{:}\Box_{\mathbf{R}} F,\ \mathbf{S} \sqsubseteq R \in \mathcal{N}_0 \subseteq \mathcal{B}$ and $\mathcal{N}_0$ did not contain $d{:}F$, then any node labelled by $d{:}F$ in $\mathcal{B}$ (if present) would be a phantom. Therefore, in order for $\mathcal{B}$ to be complete, the $\Box_i$ rule should be applied, generating a node $(k)\, d{:}F \in \mathcal{B}$. Since m and k would be siblings w.r.t. the offspring relation, k would not be a phantom in $\mathcal{B}$, therefore $k \in \mathcal{N}_0$.

    Induction Step. Let us assume that $a{:}\Box_{\mathbf{R}} F,\ a \Rightarrow_{\mathbf{S}} d,\ \mathbf{S} \sqsubseteq \mathbf{R} \in \mathcal{N}_i$. By Lemma 4, F does not contain any non-top nominal, hence $\theta_i(F) = F$ for any i.

    We distinguish two cases:

(a) $a{:}\,\Box_{\mathbf{R}}F \notin \mathcal{N}_{i-1}$. By Property P3, then, $a = b^i$ is the new nominal introduced at stage i. Therefore, $\mathcal{N}_0$ contains nodes labelled by $\theta_i^-(b^i{:}\,\Box_{\mathbf{R}}F) = \theta_i^-(b^i){:}\,\Box_{\mathbf{R}}F$ and $\theta_i^-(b^i \Rightarrow_{\mathbf{S}} d = \theta_i^-(b^i) \Rightarrow_{\mathbf{S}} \theta_i^-(d)$. Since $\mathcal{N}_0$ contains also $\mathbf{S} \sqsubseteq \mathbf{R}$ and is pseudo-saturated, $\theta_i^-(d){:}\,F \in \mathcal{N}_0$, so that $\theta_i(\theta_i^-(d)){:}\,F = d{:}\,F \in \mathcal{N}_i$.

(b) $a{:}\,\Box_{\mathbf{R}}F \in \mathcal{N}_{i-1}$. If also $a \Rightarrow_{\mathbf{S}} d \in \mathcal{N}_{i-1}$, then $d{:}\,F \in \mathcal{N}_{i-1} \subseteq \mathcal{N}_i$ by the induction hypothesis. Otherwise, $\theta_i^-(a) \Rightarrow_{\mathbf{S}} \theta_i^-(d) \in \mathcal{N}_0$. Let $a' = \theta_i^-(a)$ and $d' = \theta_i^-(d)$. By Property P4, a and a' are compatible in $\mathcal{N}_i$, therefore $a'{:}\,\Box_{\mathbf{R}}F \in \mathcal{N}_i$. Moreover, since a' occurs in $\mathcal{N}_0$, by Property P3, $a'{:}\,\Box_{\mathbf{R}}F \in \mathcal{N}_0$. Since also $a' \Rightarrow_{\mathbf{S}} d'$, $\mathbf{S} \sqsubseteq \mathbf{R} \in \mathcal{N}_0$ and $\mathcal{N}_0$ is pseudo-saturated, $d'{:}\,F \in \mathcal{N}_0$, so that also $\theta_i(d'){:}\,F = d{:}\,F \in \mathcal{N}_i$.

17. *Base.* If $(m)\, a \Rightarrow_{\mathbf{P}} b \in \mathcal{N}_0$, then m is not a phantom in $\mathcal{B}$. If also $a{:}\,\Box_{\mathbf{S}}F$, $\mathbf{P} \sqsubseteq \mathbf{R}$, $\mathsf{Trans}(\mathbf{R})$, $R \sqsubseteq S \in \mathcal{N}_0$, then the Trans_i rule has been applied, generating $(k)\, b{:}\,\Box_{\mathbf{R}}F$; k is a sibling of m, hence non-phantom too, and belongs to $\mathcal{N}_0$.

Induction Step. Let us assume that $\mathbf{P} \sqsubseteq \mathbf{R}$, $\mathsf{Trans}(\mathbf{R})$, $\mathbf{R} \sqsubseteq \mathbf{S} \in \mathcal{N}_0$, that $(n)\, a{:}\,\Box_{\mathbf{S}}F$, $(m)\, a \Rightarrow_{\mathbf{P}} d \in \mathcal{N}_i$ and at least one of n and m does not belong to $\mathcal{N}_{i-1}$ (otherwise the thesis follows from induction hypothesis). By Lemma 4, F does not contain any non-top nominal, hence $\theta_i(F) = F$ for any i.

We distinguish two cases:

(a) $a{:}\,\Box_{\mathbf{S}}F \notin \mathcal{N}_{i-1}$. By Property P3, then, $a = b^i$ is the new nominal introduced at stage i. Therefore, $\mathcal{N}_0$ contains nodes labelled by $\theta_i^-(b^i{:}\,\Box_{\mathbf{S}}F) = \theta_i^-(b^i){:}\,\Box_{\mathbf{S}}F$ and $\theta_i^-(b^i \Rightarrow_{\mathbf{P}} d) = \theta_i^-(b^i) \Rightarrow_{\mathbf{P}} \theta_i^-(d)$. Since $\mathcal{N}_0$ is pseudo-saturated, $\theta_i^-(d){:}\,\Box_{\mathbf{R}}F \in \mathcal{N}_0$, so that $\theta_i(\theta_i^-(d)){:}\,\Box_{\mathbf{R}}F = d{:}\,\Box_{\mathbf{R}}F \in \mathcal{N}_i$.

(b) $a{:}\,\Box_{\mathbf{S}}F \in \mathcal{N}_{i-1}$. If $a \Rightarrow_{\mathbf{P}} d \notin \mathcal{N}_{i-1}$, then $\theta_i^-(a) \Rightarrow_{\mathbf{P}} \theta_i^-(d) \in \mathcal{N}_0$. Let $a' = \theta_i^-(a)$ and $d' = \theta_i^-(d)$. By Property P4, a and a' are compatible in $\mathcal{N}_i$, therefore $a'{:}\,\Box_{\mathbf{S}}F \in \mathcal{N}_i$. Moreover, since a' occurs in $\mathcal{N}_0$, by Property P3, $a'{:}\,\Box_{\mathbf{S}}F \in \mathcal{N}_0$. Since also $a' \Rightarrow_{\mathbf{P}} d' \in \mathcal{N}_0$ and $\mathcal{N}_0$ is pseudo-saturated, $d'{:}\,\Box_{\mathbf{R}}F \in \mathcal{N}_0$, so that also $\theta_i(d'){:}\,\Box_{\mathbf{R}}F = d{:}\,\Box_{\mathbf{R}}F \in \mathcal{N}_i$. $\Box$

The construction of a model of $\mathcal{N}_{\mathcal{B}}^{\infty}$ is here substantially different from the corresponding one in [7], and is inspired by [11]. In order to simplify the presentation, an intermediate result is stated and proved next, based on the following definition.

Definition D2. *Let $\mathcal{B}$ be a complete and open branch. For every relation symbol R occurring in $\mathcal{B}$, the following notions are defined (in the first two items the branch $\mathcal{B}$ is left implicit so as to lighten the notation):*

1. $R_{\subseteq} = \{\langle a,b\rangle \mid a \Rightarrow_{\mathbf{S}} b \in \mathcal{N}_{\mathcal{B}}^{\infty}$ *for some* $\mathbf{S} \sqsubseteq R \in \mathcal{B}\}$, *and* $R_{\subseteq}^- = \{\langle a,b\rangle \mid \langle b,a\rangle \in R_{\subseteq}\}$.

   Moreover, $\mathbf{R}_{\subseteq}$ *is an abbreviation for* $R_{\subseteq}$ *if* $\mathbf{R}$ *is a positive relation, otherwise it stands for* $R_{\subseteq}^-$.

2. $(\mathbf{R}_{\subseteq})^+$ *is the transitive closure of* $\mathbf{R}_{\subseteq}$.

3. $\rho_{\mathcal{B}}$ *is the function on relation symbols defined as follows:*

$$\rho_{\mathcal{B}}(R) = \begin{cases} (R_{\subseteq})^+ & \textit{if } \mathsf{Trans}(R) \in \mathcal{B} \\ R_{\subseteq} \cup \{(\mathbf{S}_{\subseteq})^+ \mid \mathbf{S} \sqsubseteq R \in \mathcal{B} \textit{ and } \mathsf{Trans}(\mathbf{S}) \in \mathcal{B}\} & \textit{otherwise} \end{cases}$$

   $\rho_{\mathcal{B}}(R^-)$ *stands for* $\{\langle a,b\rangle \mid \langle b,a\rangle \in \rho_{\mathcal{B}}(R)\}$.

Below, we shall moreover use the notation $\mathsf{Inv}(\mathbf{R})$ to denote R^- if $\mathbf{R} = R$ is a positive relation; otherwise, if $\mathbf{R} = R^-$, then $\mathsf{Inv}(\mathbf{R}) = R$.

Lemma C. *If $\mathcal{B}$ is a complete and open branch, then:*

1. *for every R such that $\mathsf{Trans}(R) \in \mathcal{B}$, $\rho_{\mathcal{B}}(R)$ is a transitive relation;*

2. *for every $\mathbf{S} \sqsubseteq R \in \mathcal{B}$, $\mathbf{S}_{\subseteq} \subseteq R_{\subseteq}$;*

3. *for every relation symbol R and nominals a, b, if $\mathsf{Trans}(R) \in \mathcal{B}$, then $\langle a,b\rangle \in \rho_{\mathcal{B}}(R)$ if and only if there are nominals $c_0 = a, c_1, \ldots, c_n, c_{n+1} = b$ and relations $\mathbf{P}_0, \ldots, \mathbf{P}_n$, for $n \geq 0$, such that $\mathbf{P}_i \sqsubseteq R \in \mathcal{B}$ and $c_i \Rightarrow_{\mathbf{P}_i} c_{i+1} \in \mathcal{N}_{\mathcal{B}}^{\infty}$, for all $i = 0 \ldots n$.*

4. for every $\mathbf{S} \sqsubseteq R \in \mathcal{B}$*,* $\rho_{\mathcal{B}}(\mathbf{S}) \subseteq \rho_{\mathcal{B}}(R)$*.*

5. For every relation $\mathbf{R}$ *and nominals* a, b*, if* $\langle a, b\rangle \in \rho_{\mathcal{B}}(\mathbf{R})$ *then one of the following cases holds:*

- $a \Rightarrow_{\mathbf{S}} b \in \mathcal{N}_{\mathcal{B}}^{\infty}$ *for some* $\mathbf{S} \sqsubseteq \mathbf{R} \in \mathcal{B}$*;*
- *there exist relations* $\mathbf{P}, \mathbf{P}_0, \ldots, \mathbf{P}_n$ *and nominals* $c_0 = a, \ldots, c_{n+1} = b$ $(n \geq 0)$*, such that* $\mathsf{Trans}(\mathbf{P})$*,* $\mathbf{P} \sqsubseteq \mathbf{R} \in \mathcal{B}$ *and, for all* $i = 0 \ldots n$*,* $\mathbf{P}_i \sqsubseteq \mathbf{P} \in \mathcal{B}$ *and* $c_i \Rightarrow_{\mathbf{P}_i} c_{i+1} \in \mathcal{N}_{\mathcal{B}}^{\infty}$*.*

Proof. The first item follows directly from the definition of $\rho_{\mathcal{B}}$.

2. Let us assume that $\mathbf{S} \sqsubseteq R \in \mathcal{B}$ and $\langle a, b\rangle \in \mathbf{S}_{\subseteq}$. Then $a \Rightarrow_{\mathbf{P}} b \in \mathcal{N}_{\mathcal{B}}^{\infty}$ for some $\mathbf{P} \sqsubseteq \mathbf{S} \in \mathcal{B}$. Since $\mathbf{P} \sqsubseteq \mathbf{S}$ and $\mathbf{S} \sqsubseteq R$ are both in $\mathcal{B}$ and $\mathcal{B}$ is complete, also $\mathbf{P} \sqsubseteq R$ is in $\mathcal{B}$. Therefore $\langle a, b\rangle \in R_{\subseteq}$.

3. Let us assume that that there are nominals $c_0 = a, c_1, \ldots, c_n, c_{n+1} = b$ and relations $\mathbf{P}_0, \ldots, \mathbf{P}_n$ such that $\mathbf{P}_i \sqsubseteq R \in \mathcal{B}$ and $c_i \Rightarrow_{\mathbf{P}_i} c_{i+1} \in \mathcal{N}_{\mathcal{B}}^{\infty}$, for all $i = 0 \ldots n$. Then, $\langle c_i, c_{i+1}\rangle \in R_{\subseteq}$, by definition. If moreover $\mathsf{Trans}(R) \in \mathcal{B}$, then $\rho_{\mathcal{B}}(R) = (R_{\subseteq})^+$, therefore $\langle a, b\rangle \in \rho_{\mathcal{B}}(R)$.

   For the other direction, let us assume that $\mathsf{Trans}(R) \in \mathcal{B}$ and $\langle a, b\rangle \in \rho_{\mathcal{B}}(R) = (R_{\subseteq})^+$. Then there are nominals $c_0 = a, c_1, \ldots, c_n, c_{n+1} = b$, for $n \geq 0$, such that $\langle c_i, c_{i+1}\rangle \in R_{\subseteq}$ for all $i = 0 \ldots n$. For each such i, $c_i \Rightarrow_{\mathbf{P}_i} c_{i+1} \in \mathcal{N}_{\mathcal{B}}^{\infty}$ for some $\mathbf{P}_i \sqsubseteq R$.

4. Let us assume that $\mathbf{S} \sqsubseteq R \in \mathcal{B}$. We distinguish the following cases:

   (a) Both $\mathsf{Trans}(\mathbf{S})$ and $\mathsf{Trans}(R)$ are in $\mathcal{B}$. Then $\rho_{\mathcal{B}}(\mathbf{S}) \subseteq \rho_{\mathcal{B}}(R)$ follows from item 2 and the definition of $\rho_{\mathcal{B}}$.

   (b) If $\mathsf{Trans}(R) \notin \mathcal{B}$ and $\mathsf{Trans}(\mathbf{S}) \in \mathcal{B}$, then $\rho_{\mathcal{B}}(\mathbf{S}) \subseteq \rho_{\mathcal{B}}(R)$ follows directly from the definition of $\rho_{\mathcal{B}}$.

   (c) Let us finally consider the case where $\mathsf{Trans}(\mathbf{S}) \notin \mathcal{B}$ and $\mathsf{Trans}(R) \in \mathcal{B}$, and let us assume that $\langle a, b\rangle \in \rho_{\mathcal{B}}(\mathbf{S})$.
   If $\langle a, b\rangle \in \mathbf{S}_{\subseteq}$, then $\langle a, b\rangle \in R_{\subseteq} \subseteq \rho_{\mathcal{B}}(R)$ by item 2.
   If $\langle a, b\rangle \notin \mathbf{S}_{\subseteq}$, then $\langle a, b\rangle \in \rho_{\mathcal{B}}(\mathbf{P})$ for some $\mathbf{P}$ such that $\mathbf{P} \sqsubseteq \mathbf{S}$, $\mathsf{Trans}(\mathbf{P}) \in \mathcal{B}$.
   By item 3, there are nominals $c_0 = a, \ldots, c_n, c_{n+1} = b$ and relations $\mathbf{P}_0, \ldots, \mathbf{P}_n$ such that $\mathbf{P}_i \sqsubseteq \mathbf{P} \in \mathcal{B}$ and $\mathcal{N}_{\mathcal{B}}^{\infty}$ contains $c_i \Rightarrow_{\mathbf{P}_i} c_{i+1}$ for all $i = 0 \ldots n$. Since $\mathbf{P}_i \sqsubseteq \mathbf{P}$, $\mathbf{P} \sqsubseteq \mathbf{S}$ and $\mathbf{S} \sqsubseteq R$ are all in $\mathcal{B}$, the branch also contains $\mathbf{P}_i \sqsubseteq R$ for all $i = 0 \ldots n$. Therefore, $\langle c_i, c_{i+1}\rangle \in R_{\subseteq}$ for all $i = 0 \ldots n$ and $\langle a, b\rangle \in (R_{\subseteq})^+ = \rho_{\mathcal{B}}(R)$.

5. In order to make the distinction between positive and negative relations clearer, we distinguish the two cases.

   (a) $\mathbf{R}$ is a positive relation. If $\mathsf{Trans}(R) \in \mathcal{B}$, then the second case holds, following from item 3, taking $\mathbf{P} = \mathbf{P}_i = R$ for all $i = 0, \ldots, n$ (since $\mathcal{B}$ is complete, it contains $R \sqsubseteq R$).
   Let us assume that $\mathsf{Trans}(R) \notin \mathcal{B}$ and $\langle a, b\rangle \in \rho_{\mathcal{B}}(R)$. If $\langle a, b\rangle \in R_{\subseteq}$ then the first case holds because $R_{\subseteq} \subseteq \rho_{\mathcal{B}}(R)$. If $\langle a, b\rangle \notin R_{\subseteq}$, then $\langle a, b\rangle \in \rho_{\mathcal{B}}(\mathbf{P})$ for some $\mathbf{P}$ such that $\mathbf{P} \sqsubseteq R \in \mathcal{B}$ and $\mathsf{Trans}(\mathbf{P}) \in \mathcal{B}$. Therefore the second case holds, following from item 3.

   (b) If $\mathbf{R} = R^-$, then $\langle a, b\rangle \in \rho_{\mathcal{B}}(\mathbf{R})$ if and only if $\langle b, a\rangle \in \rho_{\mathcal{B}}(R)$. From the above case it follows that one of the following cases holds:

   - $b \Rightarrow_{\mathbf{S}} a \in \mathcal{N}_{\mathcal{B}}^{\infty}$ for some $\mathbf{S} \sqsubseteq R \in \mathcal{B}$. Since $\mathbf{S} \sqsubseteq R = \mathsf{Inv}(\mathbf{S}) \sqsubseteq R^-$ and $b \Rightarrow_{\mathbf{S}} a = a \Rightarrow_{\mathsf{Inv}(\mathbf{S})} b$, the first case holds.
   - there exist relations $\mathbf{P}, \mathbf{P}_0, \ldots, \mathbf{P}_n$ and nominals $c_0 = a, \ldots, c_{n+1} = b$, such that $\mathbf{P} \sqsubseteq R$, $\mathsf{Trans}(\mathbf{P}) \in \mathcal{B}$ and, for all $i = 0 \ldots n$, $\mathbf{P}_i \sqsubseteq \mathbf{P} \in \mathcal{B}$ and $c_{i+1} \Rightarrow_{\mathbf{P}_i} c_i \in \mathcal{N}_{\mathcal{B}}^{\infty}$. Then the second case holds because $\mathbf{P} \sqsubseteq R = \mathsf{Inv}(\mathbf{P}) \sqsubseteq R^-$, $\mathsf{Trans}(\mathbf{P}) = \mathsf{Trans}(\mathsf{Inv}(\mathbf{P}))$, $\mathbf{P}_i \sqsubseteq \mathbf{P} = \mathsf{Inv}(\mathbf{P}_i) \sqsubseteq \mathsf{Inv}(\mathbf{P})$ and $c_{i+1} \Rightarrow_{\mathbf{P}_i} c_i = c_i \Rightarrow_{\mathsf{Inv}(\mathbf{P}_i)} c_{i+1}$. □

Lemma 13. *If* $\mathcal{B}$ *is a complete and open branch, then the possibly infinite set* $\mathcal{N}_{\mathcal{B}}^{\infty}$ *has a model.*

Proof. Let $\mathcal{M} = \langle W, \rho, N, I\rangle$ be defined as follows:

- W is the set of all the nominals occurring in $\mathcal{N}_{\mathcal{B}}^{\infty}$;

- $\rho = \rho_{\mathcal{B}}$.
- $N(a) = a$ for every nominal a;
- for any $p \in \mathsf{PROP}$, $p \in I(a)$ if and only if $a\colon p$ is the label of some node in $\mathcal{N}^{\infty}_{\mathcal{B}}$;

The fact that $\mathcal{M}$ is a model of the set of assertions in $\mathcal{B}$ follows from Lemma C.

Next we prove that, for every $a\colon F \in \mathcal{N}^{\infty}_{\mathcal{B}}$, $\mathcal{M}_a \models F$. The proof is by induction on F. All cases are straightforward consequences of the definition of $\mathcal{M}$, Lemma 12 and the fact that every blockable node has its witness(es) in $\mathcal{N}^{\infty}_{\mathcal{B}}$, except for the case where $F = \Box_{\mathbf{R}}G$, whose treatment is shown below.

We shall use the notation $F_1, \ldots, F_n \Rightarrow^{k(\mathcal{R})} F$ to mean that from the fact that $F_1, \ldots, F_n \in \mathcal{N}^{\infty}_{\mathcal{B}}$ it can be inferred that $F \in \mathcal{N}^{\infty}_{\mathcal{B}}$, because, by Lemma 12, $\mathcal{N}^{\infty}_{\mathcal{B}}$ satisfies item k of Definition 14 (corresponding to the expansion rule $\mathcal{R}$).

($\Box_{\mathbf{R}}$) Let us assume that $a\colon \Box_{\mathbf{R}}G \in \mathcal{N}^{\infty}_{\mathcal{B}}$. It must be shown that $\mathcal{M}_b \models G$ for every b such that $\langle a, b\rangle \in \rho_{\mathcal{B}}(\mathbf{R})$.

By item 5 of Lemma C, if $\langle a, b\rangle \in \rho(\mathbf{R})$, then one of the cases that follows holds. For each of them we show that $b\colon G \in \mathcal{N}^{\infty}$; therefore, by the induction hypothesis $\mathcal{M}_b \models G$.

1. $a \Rightarrow_{\mathbf{S}} b \in \mathcal{N}^{\infty}_{\mathcal{B}}$ for some $\mathbf{S} \sqsubseteq \mathbf{R} \in \mathcal{B}$. Then $b\colon G \in \mathcal{N}^{\infty}$ because:

$$a\colon \Box_{\mathbf{R}}G,\ a \Rightarrow_{\mathbf{S}} b,\ \mathbf{S} \sqsubseteq \mathbf{R} \quad \Rightarrow^{13(\Box_i)} \quad b\colon G$$

2. there exist relations $\mathbf{P}, \mathbf{P}_0, \ldots, \mathbf{P}_n$ and nominals $c_0 = a, \ldots, c_{n+1} = b$ ($n \geq 0$), such that $\mathsf{Trans}(\mathbf{P})$, $\mathbf{P} \sqsubseteq \mathbf{R} \in \mathcal{B}$ and, for all $i = 0 \ldots n$, $\mathbf{P}_i \sqsubseteq \mathbf{P} \in \mathcal{B}$ and $c_i \Rightarrow_{\mathbf{P}_i} c_{i+1} \in \mathcal{N}^{\infty}_{\mathcal{B}}$. Then:

$$\begin{array}{lll}
a\colon \Box_{\mathbf{R}}G,\ a \Rightarrow_{\mathbf{P}_0} c_1,\ \mathbf{P}_0 \sqsubseteq \mathbf{P}, \mathsf{Trans}(\mathbf{P}),\ \mathbf{P} \sqsubseteq \mathbf{R} & \Rightarrow^{14(\mathsf{Trans}_i)} & c_1\colon \Box_{\mathbf{P}}G \\
c_1\colon \Box_{\mathbf{P}}G,\ c_1 \Rightarrow_{\mathbf{P}_1} c_2,\ \mathbf{P}_1 \sqsubseteq \mathbf{P}, \mathsf{Trans}(\mathbf{P}),\ \mathbf{P} \sqsubseteq \mathbf{R} & \Rightarrow^{14(\mathsf{Trans}_i)} & c_2\colon \Box_{\mathbf{P}}G \\
\ldots & \Rightarrow^{14(\mathsf{Trans}_i)} & c_n\colon \Box_{\mathbf{P}}G \\
c_n\colon \Box_{\mathbf{P}}G,\ c_n \Rightarrow_{\mathbf{P}_n} b,\ \mathbf{P}_n \sqsubseteq \mathbf{P} & \Rightarrow^{13(\Box_i)} & b\colon G
\end{array}$$

□

Completeness can finally be proved using Lemma 13 like in [7].

6 Concluding remarks

This work presents a satisfiability decision procedure for hybrid formulae in $\mathsf{HL}_m(@, \downarrow, \mathsf{E}, \Diamond^-, \mathsf{Trans}, \sqsubseteq) \setminus \Box\!\downarrow\!\Box$. Transitivity and relation inclusion assertions are treated by expansion rules which are essentially reformulations of the analogous rules presented in [11, 12, 13, 14]. The main result of this work is proving that they can be added to a calculus dealing also with (restricted occurrences of) the binder, maintaining termination, beyond soundness and completeness.

It is worth pointing out that, like in [5, 4, 8, 7], the calculus presented in this work treats nominal equalities by means of substitution, and this is essential to ensure a key property of the calculus, used to prove both termination and completeness: any universal formula occurring in a tableau branch is a subformula of the top formula, therefore a branch cannot contain an unbounded number of universal formulae.

Other works have addressed the issue of representing frame properties and/or relation hierarchies in tableau calculi for binder-free hybrid logic (for instance, [3, 13, 14]). The maybe richer calculus of this kind is [14], that considers graded and global modalities, reflexivity, transitivity and role hierarchies. The converse modalities are however missing, and inverse relations are not allowed.

The possibility of adding graded modalities to the calculus presented in this work is an interesting but hard issue. As a matter of fact, whether restricted occurrences of the binder can coexist with graded modalities in a decidable hybrid logic is an open question.

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