Dipartimento di Informatica e Automazione
Via della Vasca Navale, 79
00146 Roma, Italy

# On the Area Requirements of Euclidean Minimum Spanning Trees

Patrizio Angelini[1], Till Bruckdorfer[2], Marco Chiesa[1],
Fabrizio Frati[1,3], Michael Kaufmann[2], and Claudio Squarcella[1]

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(1) Università Roma Tre, Italy

(2) Universität Tübingen, Germany

(3) EPFL Lausanne, Switzerland

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## ABSTRACT

In their seminal paper on Euclidean minimum spanning trees *[Discrete & Computational Geometry, 1992]*, Monma and Suri proved that any tree of maximum degree 5 admits a planar embedding as a Euclidean minimum spanning tree. Their algorithm constructs embeddings with exponential area; however, the authors conjectured that $c^n \times c^n$ area is sometimes required to embed an $n$-vertex tree of maximum degree 5 as a Euclidean minimum spanning tree, for some constant $c > 1$. In this paper, we prove the first exponential lower bound on the area requirements for embedding trees as Euclidean minimum spanning trees.

A *Euclidean minimum spanning tree* (MST) of a set $P$ of points in the plane is a tree with a vertex in each point of $P$ and with minimum total edge length. Euclidean minimum spanning trees have several applications in computer science and hence they have been deeply investigated from a theoretical point of view. To cite a few major results, optimal $\Theta(n \log n)$-time algorithms are known to compute an MST of a set of points and it is $\mathcal{NP}$-hard to compute an MST with maximum degree bounded by 2, 3, or 4 [4, 6, 13], while polynomial-time algorithms exist [1, 2, 8, 11] to compute MST with maximum degree bounded by 2, 3, or 4 and total edge length within a constant factor from the optimal one.

An *MST embedding* of a tree $T$ is a plane embedding of $T$ such that the MST of the points where the vertices of $T$ are drawn coincides with $T$. In this paper we consider the problem of constructing MST embeddings of trees. Several results are known related to such a problem. No tree having a vertex of degree at least 7 admits an MST embedding. Further, deciding whether a tree with degree 6 admits an MST embedding is $\mathcal{NP}$-hard [3]. However, restricting the attention to trees of degree 5 is not a limitation since: (i) every planar point set has an MST with maximum degree 5 [12], and (ii) every tree of maximum degree 5 admits an MST embedding in the plane [12].

Euclidean MST embeddings have also been considered as a subtopic of proximity drawings [14], where adjacent vertices have to be placed "close" to each other. From the various applications, different measures for closeness have been introduced and evaluated, different graph classes that can be realized under the specific closeness measure have been studied together with the area/volume that is needed for their realization. Prominent examples are Delaunay triangulations, Gabriel drawings, nearest-neighbor drawings, $\beta$-drawings, and many more. The MST constraints can be formulated as closeness conditions with respect to pairs of vertices, either adjacent or non-adjacent.

Monma and Suri's proof [12] that every tree of maximum degree 5 admits an MST embedding in the plane is a strong combinatorial result; on the other hand, their algorithm for constructing MST embeddings seems to be useless in practice, since the constructed embeddings have $2^{\Theta(k^2)}$ area for trees of height $k$ (hence, in the worst case the area requirement of such drawings is $2^{\Theta(n^2)}$). However, Monma and Suri conjectured that there exist trees of maximum degree 5 that require $c^n \times c^n$ area in *any* MST embedding, for some constant $c > 1$. The problem of determining whether or not the area upper bound for MST embeddings of trees can be improved to polynomial is reported also in [3, 7, 10]. Recently, MST embeddings in polynomial area have been proven to exist for trees with maximum degree 4 [5, 9].

In this paper, we prove that there exist $n$-vertex trees of maximum degree 5 requiring $2^{\Omega(n)}$ area in any MST embedding. Our lower bound is achieved by considering an $n$-vertex tree $T^*$, shown in Fig. 1, composed of a degree-5 *complete tree* $T_c$ with a constant number of vertices and of a set of degree-5 *caterpillars*, each one attached to a distinct leaf of $T_c$. The argument goes in two steps: For the first step, we walk down the tree $T^*$, starting from the root. The route is chosen so that the angles adjacent to the edges are narrowing at each step. The key observation here is a lemma relating the size of two consecutive angles adjacent to an edge. At the leaves of the complete tree $T_c$, where the caterpillars start, the angles incident to an end-vertex of the backbone of at least one of the caterpillars must be very small, that is, between $60^\circ$ and $61^\circ$. Using this as a starting point, we prove that each angle incident to a vertex of the caterpillar is either very small, that is, between $60^\circ$ and $61^\circ$, or is very large, that is, between $89.5^\circ$ and $90.5^\circ$. As a

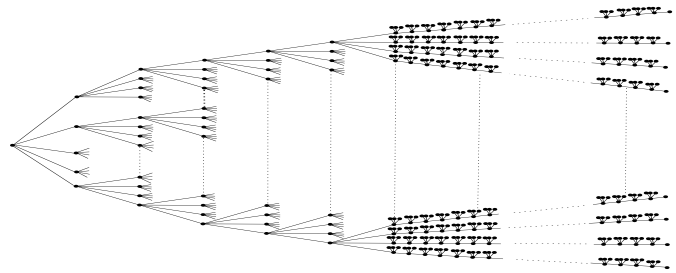


Figure 1: A tree $T^*$ requiring $2^{\Omega(n)}$ area in any MST embedding.

consequence, we show that when walking down along the backbone of the caterpillar, the lengths of the edges decrease exponentially along the caterpillar. Since the backbone has a linear number of edges, we obtain the claimed area bound.

The paper is organized as follows. In Sect. 1 we give some definitions and preliminaries; in Sect. 2 we give some geometric lemmata; in Sect. 3 we argue about the angles and the edge lengths of the MST embeddings of $T^*$; in Sect. 4 we prove the area lower bound; finally, we conclude with some remarks and a conjecture in Sect. 5.

# 1 Preliminaries

A *rooted tree* is a tree with one distinguished vertex, called *root*. The *depth* of a vertex in a rooted tree is its distance from the root, that is, the number of edges in the path from the root to the vertex. The *height* of a rooted tree is the maximum depth of one of its vertices. A *complete tree* is such that every path from the root to a leaf has the same number of vertices and every vertex has the same degree. A *caterpillar* is a tree such that removing the leaves yields a path, called the *backbone* of the caterpillar.

A *minimum spanning tree* MST of a planar point set $P$ is a tree spanning $P$ with minimum total edge length. Given a tree $T$, an *MST embedding* of $T$ is a straight-line drawing of $T$ such that the MST of the points where the vertices of $T$ are drawn is isomorphic to $T$. The *area* of an MST embedding is the area of a rectangle enclosing such an embedding. The concept of area of an MST embedding only makes sense once a *resolution rule* is fixed, i.e., a rule that does not allow vertices to be arbitrarily close (*vertex resolution rule*), or edges to be arbitrarily short (*edge resolution rule*). Without any of such rules, one could just construct MST embeddings with arbitrarily small area. In the following we will hence suppose that any two vertices have distance at least one unit. Then, in order to prove that an $n$-vertex tree $T$ requires $f(n)$ area in any MST embedding, it suffices to prove that the ratio between the longest and the shortest edge of any MST embedding is $f(n)$, and that both dimensions have at least constant size.

Consider any MST embedding of a tree $T$ rooted at a node $r$. The *clockwise path* $Cl(u)$ of a vertex $u \neq r$ of $T$ is the path $v_0, v_1, \ldots, v_k$ such that $v_0 = u$, $(v_i, v_{i+1})$ is the edge following the edge from $v_i$ to its parent in the clockwise order of the edges incident to $v_i$, for $i = 0, \ldots, k-1$, and $v_k$ is a leaf. The *counterclockwise path* $Ccl(u)$ of a vertex $u \neq r$ of $T$ is defined analogously. Denote by $d(a, b)$ the Euclidean distance between two vertices $a$ and $b$ (or between two points $a$ and $b$) and by $|e|$ the length of an edge $e$. Further, $k(c, r)$ denotes the circle centered at a point $c$ and having radius $r$.

Next, we define an $n$-vertex tree $T^*$ that requires $\Omega(2^n)$ area in any MST embedding.

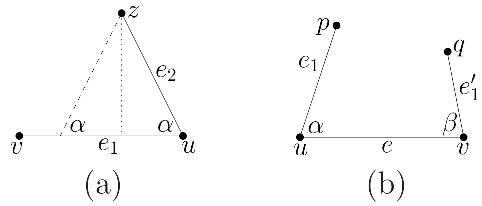



Figure 2: (a) Illustration for the proof of Lemma 5. (b) The setting for Lemma 6.

Let $T_c$ be a complete tree of height six and degree five. Let $r$ be the root of $T_c$. Augment $T_c$ by inserting a degree-five caterpillar at each leaf of $T_c$. That is, for each leaf $l$ of $T_c$, insert a caterpillar $C_l$ whose every non-leaf vertex has degree five, such that $l$ is an end-vertex of the backbone of $C_l$, the parent of $l$ in $T_c$ is a leaf of $C_l$, and $C_l$ and $T_c$ do not share any other vertex. The resulting tree, shown in Fig. 1, is denoted by $T^*$.

## 2 Geometric Lemmata

In this section we give some properties for MST embeddings. The first four lemmata are well-known and can be easily proven.

**Lemma 1** *A straight-line drawing of a tree $T$ is an MST embedding of $T$ if and only if, for each pair of vertices $u$ and $v$ of $T$, $d(u,v) \geq |e|$, for each edge $e$ in the path connecting $u$ and $v$ in $T$.*

**Lemma 2** *In any MST embedding of a tree, any angle between two adjacent segments is at least $60°$.*

**Lemma 3** *Consider any MST embedding $\Gamma$ of a tree $T$. Consider any subtree $T'$ of $T$. Then, $\Gamma$ restricted to the vertices and edges of $T'$ is an MST embedding of $T'$.*

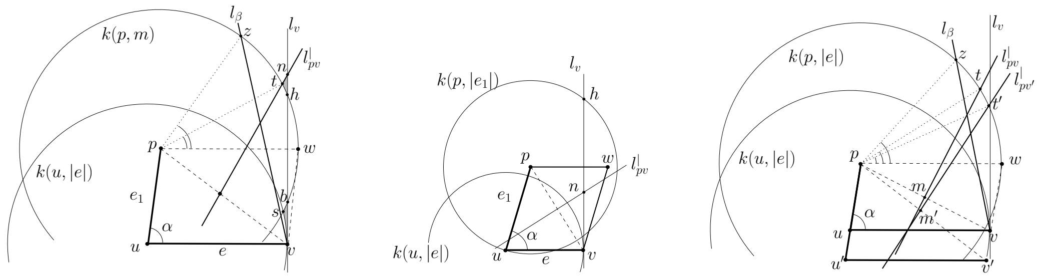
**Lemma 4** *Any MST embedding of a tree $T$ is planar.*

The next lemma bounds the length of an edge in an MST embedding in terms of the length of an adjacent edge and of the size of the angle between them.

**Lemma 5** *Let $e_1$ and $e_2$ be two edges consecutively incident to the same vertex and let $\alpha \leq 90°$ be the angle they form. Then, $2|e_1|\cos(\alpha) \leq |e_2| \leq \frac{|e_1|}{2\cos(\alpha)}$.*

**Proof:** Refer to Fig. 2(a). Let $e_1 = (u,v)$ and $e_2 = (u,z)$. If $|e_1| < 2|e_2|\cos\alpha$, then $|(v,z)| < |(u,z)|$, thus contradicting Lemma 1. Hence, $|e_1| \geq 2|e_2|\cos\alpha$. Analogously, $|e_2| \geq 2|e_1|\cos\alpha$. □

Consider an edge $e = (u,v)$ in an MST embedding of a tree $T$. See Fig. 2(b). Let $e_1 = (u,p)$ be the edge following $e$ in the counterclockwise order of the edges incident to $u$ and $e'_1 = (v,q)$ be the edge following $e$ in the clockwise order of the edges incident to $v$. Let $\alpha$ ($\beta$) be the angle defined by a counterclockwise (resp. clockwise) rotation of $e$ around $u$ (resp. around $v$) bringing $e$ to coincide with $e_1$ (resp. with $e'_1$). Let $\phi = \frac{|e|}{|e_1|}$. The next lemma, that establishes a strong lower bound on $\beta$ provided that $\alpha$ is sufficiently small, is one of our main tools for the remainder of the paper.



(a) (b) (c)

Figure 3: (a) Illustration for the proof of Lemma 6. The lower and upper shaded regions are respectively $R_1$ and $R_2$ where $q$ can lie. (b) When $\phi \leq 1$, region $R_2$ does not exist. (c) When $\phi > 1$, region $R_2$ is always avoided.

**Lemma 6** *Suppose that $\alpha \leq 80°$. Then, $\beta \geq 120° - \alpha/2$.*

**Proof:** First, we determine restrictions on the region where $q$ lies, once the drawings of $e$ and $e_1$ are fixed. Refer to Fig. 3(a). By Lemma 1, $d(q,u) \geq d(u,v)$ holds. Then, $q$ is outside $k(u,|e|)$. Still by Lemma 1, $d(p,q) \geq d(p,u)$ and $d(p,q) \geq d(u,v)$ hold. Then, $q$ is outside $k(p,m)$, where $m = \max\{|e|,|e_1|\}$. Again by Lemma 1, $d(p,q) \geq d(v,q)$ holds. Denote by $l^{|}_{pv}$ the line orthogonal to $\overline{pv}$ passing through the midpoint of $\overline{pv}$; then, $q$ is in the half-plane delimited by $l^{|}_{pv}$ and not containing $p$. Denote by $l_\beta$ the line passing through $v$ and creating with $\overline{uv}$ the angle $120^{\circ} - \frac{\alpha}{2}$. Suppose, w.l.o.g. up to a rotation, a reflection, and a translation of the drawing, that $e$ is horizontal, that $u$ is at point $(0,0)$, that $v$ is to the right of $u$, and that both $p$ and $q$ are above the horizontal line through $u$ and $v$. We can suppose that $q$ is to the left of the vertical line $l_v$ through $v$, since otherwise $\beta \geq 90° \geq 120° - \alpha/2$, where the last inequality holds by Lemma 2, and there is nothing to prove.

Second, we discuss about the intersections of $k(p,m)$ with $l_v$. The distance from $p$ to $l_v$ is less than $|e|$, because $p$ is to the right of the vertical line through $u$, given that $\alpha \leq 80°$. It follows that $k(p,m)$ has exactly two intersections with $l_v$, given that $m \geq |e|$. Moreover both of such intersections lie not below $v$ as the distance between $p$ and $v$ is at least $m$, by Lemma 1, and hence the distance between $p$ and any point of $l_v$ below $v$ is strictly greater than $m$, while $k(p,m)$ has radius exactly $m$. Denote by $h$ the highest of such two intersection points.

Third, we prove that for any $\alpha \leq 80^{\circ}$ the region $R_2$, bounded (when existing) by $l_v$ from the right, by $k(p,m)$ from the left and by $l^{|}_{pv}$ from above, either does not exist or falls on the right of the line $l_\beta$. We distinguish two cases, based on whether $\phi \leq 1$, that is, $|e_1| \geq |e|$, or not.

In the former case, consider a segment $\overline{vw}$ parallel to $e_1$ such that $|e_1| = |\overline{vw}|$. See Fig. 3(b). Observe that $|\overline{pw}| = |e|$. Consider triangle $\Delta(puv)$. As $|e_1| \geq |e|$, we have that $\widehat{pvu} \geq \widehat{vpu}$. Hence, the line $l^{|}_{pv}$ orthogonal to $\overline{pv}$ passing through its midpoint crosses segment $\overline{up}$. Analogously, in triangle $\Delta(pwv)$, line $l^{|}_{pv}$ crosses segment $\overline{vw}$. Hence, $l^{|}_{pv}$ cuts polygon $(u,v,w,p)$ by crossing segments $\overline{up}$ and $\overline{vw}$, which implies that point $n$ lies inside $(u,v,w,p)$. As segment $\overline{pw}$ cuts $l_v$ below $h$, we have that $|\overline{vn}| \leq |\overline{vh}|$, therefore region $R_2$ does not exist.

In the latter case, that is, when $\phi > 1$, we first prove the statement when $\phi$ assumes its maximum possible value, that is, by Lemma 5, $\phi = \frac{1}{2\cos\alpha}$; then, we extend the same statement to all the other values of $\phi$ greater than 1. Observe that, if any point of $R_2$ is to the left of $l_\beta$, then $t$ is to the left of $l_\beta$. Hence, in order to prove that $R_2$ is entirely to the right of $l_\beta$, it suffices to prove that $\widehat{uvt} \geq 120° - \alpha/2$.

Suppose that $\phi = \frac{1}{2\cos\alpha}$. Then, by Lemma 5, triangle $\triangle(uvp)$ is isosceles, with the two equal-length sides being $\overline{uv}$ and $\overline{pv}$. Hence, triangle $\triangle(pvt)$ is equilateral, as $\overline{pt}$ is a radius of $k(p,|e|)$ and $t$ is a point of $l^{|}_{pv}$. Therefore, $\widehat{uvp} = 180^{\circ} - 2\alpha$ and $\widehat{pvt} = 60^{\circ}$. Since $\widehat{uvt} = \widehat{uvp} + \widehat{pvt}$, we have that $\widehat{uvt} = 180^{\circ} - 2\alpha + 60^{\circ} = 240^{\circ} - 2\alpha$. Since $240^{\circ} - 2\alpha \geq 120^{\circ} - \frac{\alpha}{2}$ holds for any $\alpha \leq 80^{\circ}$, region $R_2$ is entirely to the right of $l_\beta$, provided that $\phi = \frac{1}{2\cos\alpha}$.

Now we extend the proof to the general case, in which $1 < \phi < \frac{1}{2\cos\alpha}$. See Fig. 3(c). Let $w$ be the right-most point of $K(p,|e|)$. Note that points $p$,$w$,$v$,and $u$ form a parallelogram. Note also that, using arguments similar to the first case $\phi \leq 1$, here line $l^{|}_{pv}$ always crosses segment $\overline{pw}$; moreover the slope of $l^{|}_{pv}$ is positive, which guarantees that $l^{|}_{pv}$ always crosses $k(p,|e|)$ at point $t$ with $0^{\circ} \leq \widehat{tpw} \leq 90^{\circ}$.

Let $z$ be the intersection point between $l_\beta$ and $k(p,|e|)$. Observe that, since $t$ lies in the first quadrant of $k(p,|e|)$, since line $l_\beta$ has a negative slope, and since line $l^{|}_{pv}$ has a positive slope, we have that $t$ is to the right of $l_\beta$, that is, $\widehat{uvt} \geq 120^{\circ} - \frac{\alpha}{2}$, if and only if $\widehat{zpw} \geq \widehat{tpw}$. Note that for the already studied case $\phi = \frac{1}{2\cos\alpha}$ the statement holds.

Hence, it suffices to show that, for any $\alpha$, given two values $\phi$ and $\phi'$ such that $\phi' < \phi$ (or equivalently $|\overline{pu'}| > |\overline{pu}|$), we have $\widehat{tpw} > \widehat{t'p'w'}$.

Suppose, w.l.o.g., that $|\overline{uv'}| = |\overline{uv}|$, that point $p$ and $p'$ coincide, and that segments $\overline{pu'}$ and $\overline{pu}$ lie on the same line. Note that $m$ and $m'$, respectively midpoints of segments $\overline{pv}$ and $\overline{p'v'}$ lie on a line which is parallel to $\overline{up}$ by construction. Moreover, $m'$ lies below $m$ and the slope of $l^{|}_{pv'}$ is smaller than the one of $l^{|}_{pv}$. This implies that $t'$ lies to the right of $t$ and therefore $\widehat{tpw} > \widehat{t'p'w'}$, hence the statement holds.

Finally, we prove the claimed lower bound for $\beta$ by defining the remaining region $R_1$ in which $q$ can lie, and showing that it always falls on the right of $l_\beta$. Region $R_1$ is bounded by $l_v$ from the right, by $k(u,|e|)$ from the left, and either by $k(p,m)$ or by $l^{|}_{pv}$ from above (depending on whether $n$ is higher or lower than $b$). Hence, such a region is a subset of the region bounded by $l_v$ from the right, by $k(u,|e|)$ from the left, and by $k(p,m)$ from above. Then, denoting by $s$ the intersection point between $k(p,m)$ and $k(u,|e|)$, we have $\beta \geq \widehat{uvs}$. Namely, the line through $v$ and $s$ has $R_1$ to its right. Then, it suffices to show that $\widehat{uvs} \geq 120° - \alpha/2$. Denote by $\gamma$ the angle $\widehat{vus}$. Then, we have $s \equiv (|e|\cos\gamma, |e|\sin\gamma)$ and $\widehat{uvs} = \frac{180° - \gamma}{2}$, where the last equality uses the fact that $|\overline{us}| = |\overline{uv}|$. Observe also that $p \equiv (|e_1|\cos\alpha, |e_1|\sin\alpha)$. We further distinguish two cases, namely the one in which $|e| \geq |e_1|$ (Case 1) and the one in which $|e_1| \geq |e|$ (Case 2).

Suppose that we are in Case 1. Then, there are two isosceles triangles, $\Delta(suv)$ and $\Delta(sup)$. Consider the triangle $\Delta(sup)$: its two equal-length sides, $\overline{us}$ and $\overline{ps}$, have length $|e|$ which is larger than the third side, which has length $|e_1|$. Thus we have $\widehat{sup} \geq 60^{\circ}$. Since $\widehat{sup} = \alpha - \gamma$ we have $\gamma \leq \alpha - 60^{\circ}$. At the same time $\widehat{uvs}$, $\widehat{usv} = \widehat{uvs}$, and $\gamma$ are the angles in $\Delta(suv)$ and therefore sum up to $180^{\circ}$. This shows that $\widehat{uvs} \geq 120° - \alpha/2$.

Case 2 is analogous to Case 1. The side lengths on the triangle $\Delta(sup)$ change: it remains an isosceles triangle, but now the two equal sized segments are $\overline{pu}$ and $\overline{ps}$, both with length $|e_1|$. The third side is shorter ($|e|$) and hence the angle $\widehat{sup}$ is again larger than $60^{\circ}$. So we can argue as in Case 1. Hence, Lemma 6 holds. □

# 3 Angles and Edge Lengths in MST Embeddings

In this section we argue about the angles and the edge lengths in each MST embedding of $T^*$. We start by providing lemmata about the complete tree $T_c$ .

**Lemma 7** *There exists two consecutive angles $\tau_1$ and $\tau_2$ incident to $r$ such that $\tau_1 + \tau_2 \leq 150°$ and $\tau_1, \tau_2 \leq 80°$.*

**Proof:** If two among the angles incident to $r$ are greater than 80°, then the other three angles sum up to less than 200°. Hence, by Lemma 2, each of them is at most 80° and any two of them sum up to at most 140°. Since two of such three angles are consecutive, the lemma follows.

If at most one among the angles incident to $r$ is greater than 80°, then the other four angles are each at most 80° and, by Lemma 2, they sum up to at most 300°. Such four angles can be subdivided into two pairs of consecutive angles; since one of such pairs has angles summing up to at most 150°, the lemma follows. □

**Lemma 8** *A vertex $u$ of $T_c$ with depth five exists such that two angles consecutively incident to $u$ and not adjacent to the edge from $u$ to its parent sum up to at most 121°.*

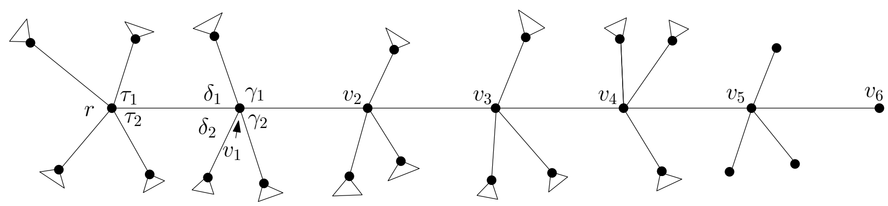



Figure 4: Tree $T_c$. In order to draw on a straight line the path composed of the edges followed in the proof of Lemma 8, the drawing of the angles is not always coherent with their actual value.

**Proof:** Refer to Fig. 4. Given an edge $(u, v)$, where both $u$ and $v$ are not leaves of $T_c$, consider the edge $(u, u_1)$ that immediately precedes $(u, v)$ in the clockwise (counterclockwise) order of the edges incident to $u$. Consider the edge $(v, v_1)$ that immediately precedes $(v, u)$ in the counterclockwise (clockwise, resp.) order of the edges incident to $v$. Then, $\widehat{u_1uv}$ is *opposite to* $\widehat{v_1vu}$ with respect to $(u, v)$. By Lemma 7, there exists two consecutive angles $\tau_1$ and $\tau_2$ incident to $r$ such that $\tau_1 + \tau_2 \leq 150°$ and $\tau_1, \tau_2 \leq 80°$. Denote by $v_1$ the neighbor of $r$ such that edge $(r, v_1)$ is adjacent to $\tau_1$ and $\tau_2$. By Lemma 6, the angles opposite to $\tau_1$ and $\tau_2$ with respect to $(r, v_1)$, say $\delta_1$ and $\delta_2$, satisfy $\delta_1 \geq 120° - \tau_1/2$ and $\delta_2 \geq 120° - \tau_2/2$. Hence, $\delta_1 + \delta_2 \geq 240° - (\tau_1 + \tau_2)/2 \geq 240° - 75° = 165°$. Denote by $\gamma_1$, $\gamma_2$, and $\gamma_3$ the angles incident to $v_1$ different from $\delta_1$ and $\delta_2$ in this clockwise order. Then, we have $\gamma_1 + \gamma_2 \leq 135°$, since $\gamma_1 + \gamma_2 + \gamma_3 \leq 195°$ and $\gamma_3 \geq 60°$. Observe that, since $\gamma_1, \gamma_2 \geq 60°$, we have $\gamma_1, \gamma_2 \leq 75°$. Next, consider the edge $(v_1, v_2)$ adjacent to $\gamma_1$ and $\gamma_2$. The two angles incident to $v_2$ and opposite to $\gamma_1$ and $\gamma_2$ sum up to at least $240° - 135°/2 = 172.5°$. Hence, any two angles consecutively incident to $v_2$ and not adjacent to $(v_1, v_2)$ sum up to at most 127.5°. Such an argument propagates along any path from $v_1$ to a leaf. Thus, there exists a path $(r, v_1, v_2, v_3, v_4, v_5, v_6)$ such that the two

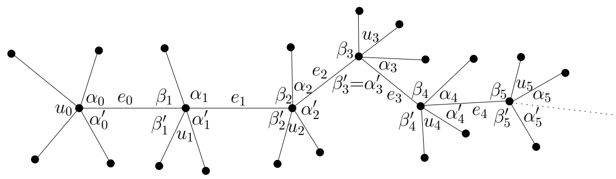



Figure 5: An embedding of $C^*$.

angles incident to $v_1$, $v_2$, $v_3$, $v_4$, and $v_5$ adjacent to edge $(v_1, v_2)$, $(v_2, v_3)$, $(v_3, v_4)$, $(v_4, v_5)$, and $(v_5, v_6)$, resp., sum up to at most 135°, 127.5°, 123.75°, 121.875°, and 120.93875°, respectively. The lemma follows with $u = v_5$. □

Consider any MST embedding of $T^*$; by Lemma 8, there exists a caterpillar $C^*$ such that one of the end-vertices $u_0$ of the backbone of $C^*$ is incident to an edge of $T_c$ that is adjacent to two angles $\alpha_0$ and $\alpha'_0$ summing up to at most $121^\circ$. Denote by $u_0, u_1, \ldots, u_k$ the vertices of the backbone of $C^*$ and by $e_i$ the backbone edge $(u_i, u_{i+1})$, for $i = 0, \ldots, k-1$. We call *outgoing angles* $\alpha_i$ *and* $\alpha'_i$ the angles adjacent to $e_i$ and incident to $u_i$; we call *incoming angles* $\beta_{i+1}$ *and* $\beta'_{i+1}$ the angles adjacent to $e_i$ and incident to $u_{i+1}$. An edge $e$ incident to $u_i$ that is not the incoming edge of $u_i$ is *in position* $j \in \{1, 2, 3, 4\}$ if $e$ is the $j$-th edge in the clockwise order of the edges incident to $u_i$ starting at $e_{i-1}$. Note that, if $e_{i+1}$ is in position 1 (resp. 4), the incoming angle $\beta_{i+1}$ and the outgoing angle $\alpha_{i+1}$ (resp. the incoming angle $\beta'_{i+1}$ and the outgoing angle $\alpha'_{i+1}$) coincide. See Fig. 5. We prove that the outgoing and the incoming angles incident to a vertex of the backbone of $C^*$ are either *small angles*, that is, between $60^\circ$ and $61^\circ$, or *large angles*, that is between $89.5^\circ$ and $90.5^\circ$. More precisely, the incoming angles are always large, while the outgoing angles are either both small or one large and one small. Indeed, observe that the outgoing angles of $u_0$ are both small by Lemma 8.

Suppose that a backbone edge $e_i$ is in position 2 or 3 and that the incoming angles of $u_i$ are at least $89.5^\circ$. By Lemma 2, each of the outgoing angles of $u_i$ is at most $61^\circ$ (recall that $e_i$ is in position 2 or 3). Then, by Lemma 6, the incoming angles of $u_{i+1}$ are at least $89.5^\circ$. Hence, if $e_i$ is in position 2 or 3 and the incoming angles of $u_i$ are at least $89.5^\circ$, the incoming angles of $u_{i+1}$ are also at least $89.5^\circ$.

If $e_i$ is in position 1 or 4, then one outgoing angle of $u_i$, say $\alpha_i$, coincides with one incoming angle of $u_i$, say $\beta_i$. Hence, $\alpha_i{=}\beta_i$ is large and no lower bound for $\beta_{i+1}$ can obtained by Lemma 6. However, we can prove that, even if $\alpha_i$ is large, angle $\beta_{i+1}$ is large, provided that the following condition is satisfied: The clockwise path $Cl(u_i)$ of $u_i$ lies in a bounded region $R_i$ that is a subset of a wedge $W_i$ with angle $1^\circ$ centered at $u_i$. We will later prove (in Lemma 14) that, if such a condition is satisfied by a node $u_i$ incident to a large outgoing angle $\alpha_i$, then $\beta_{i+1}$ is large and $Cl(u_{i+1})$ lies in a bounded region $R_{i+1}$ that is a subset of a wedge $W_{i+1}$ with angle $1^\circ$ centered at $u_{i+1}$. However, before that, we prove that such a condition is satisfied by a node $u_i$ if $\alpha_{i-1}$ is small.

Suppose, w.l.o.g. up to a rotation, a reflection, and a translation of the drawing, that $e_{i-1}$ is horizontal, with $u_i$ to the right of $u_{i-1}$, and that $e_i$ is in position 1. Denote by $e = (u_{i-1}, v)$ (by $e^* = (u_{i+1}, w)$) the edge following $e_{i-1}$ (resp. $e_i$) in the counterclockwise (resp. clockwise) order of the edges incident to $u_{i-1}$ (resp to $u_{i+1}$). Denote by $l(\underline{\alpha_i})$ (by $l(\overline{\alpha_i})$) the half-line with slope $90.5^\circ$ (resp. with slope $89.5^\circ$) starting at $u_i$. Denote by $W_i$ the closed wedge with angle $1^\circ$ delimited by $l(\underline{\alpha_i})$ and $l(\overline{\alpha_i})$. See Fig. 6.

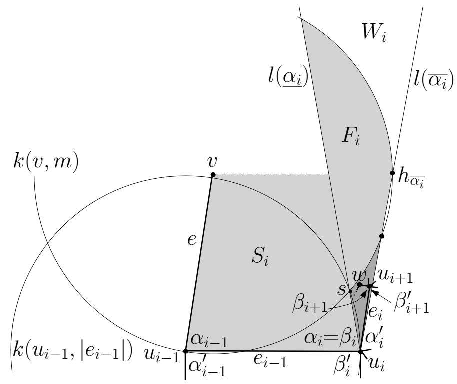



Figure 6: The setting for Lemmata 9–13. The dark-shaded region is $R_i$. To improve the readability, angles and edge lengths in the illustration do not correspond to actual angles and edge lengths.

We will bound the region in which $Cl(u_i)$ lies from the right, from the left, and from above. Let $m{=}\max\{|e|, |e_{i-1}|\}$. Concerning the bound from the left, we can prove that the intersection point $s$ of the circles $k(v,m)$ and $k(u_{i-1}, |e_{i-1}|)$ is not to the left of $l(\underline{\alpha_i})$, as stated in the following.

**Lemma 9** *Suppose that $\alpha_{i-1} \leq 61^\circ$. Then, $s$ is not to the left of $l(\underline{\alpha_i})$.*

**Proof:** The statement can be proven using exactly the same considerations as in the proof of Lemma 6. Namely, a lower bound of $120^\circ - \frac{\alpha_{i-1}}{2}$ for the slope of the line through $u_i$ and $s$ can be computed exactly as in Lemma 6. Since $\alpha_{i-1} \leq 61^\circ$, the statement follows. □

We continue with the bound from the right.

**Lemma 10** *Suppose that $\beta'_i \geq 89.5^\circ$. Then vertex $u_{i+1}$ is not to the right of $l(\overline{\alpha_i})$.*

**Proof:** By Lemma 2 the three angles incident to $u_i$ and different from $\beta_i$ and $\beta'_i$ sum up to at least $180^\circ$. The lemma follows by the assumption that $\beta'_i \geq 89.5^\circ$. □

To derive the bound from above, we prove that $k(v,m)$ intersects $l(\overline{\alpha_i})$ twice and we argue about the distance between $u_i$ and the highest intersection point $h_{\overline{\alpha_i}}$ of $k(v,m)$ with $l(\overline{\alpha_i})$.

**Lemma 11** *Suppose that $\alpha_{i-1} \leq 61^\circ$. Then, $k(v,m)$ intersects $l(\overline{\alpha_i})$ twice.*

**Proof:** We prove that $l(\overline{\alpha_i})$ intersects $k(v,m)$ twice. Suppose, w.l.o.g. up to a translation of the coordinate system that $u_{i-1}$ has coordinates $(0,0)$. Then $k(v,m)$ has equation $(y - |e|\sin\alpha_{i-1})^2 + (x - |e|\cos\alpha_{i-1})^2 = m^2$ and $l(\overline{\alpha_i})$ has equation $y = \tan 89.5^\circ(x - |e_{i-1}|)$. Substituting the second equation into the first one, we get that the $x$-coordinates of the intersections of $k(v,m)$ and $l(\overline{\alpha_i})$ satisfy $x^2\tan^2 89.5^\circ + |e_{i-1}|^2\tan^2 89.5^\circ + |e|^2\sin^2\alpha_{i-1} - 2|e_{i-1}|x\tan^2 89.5^\circ - 2|e|x\tan 89.5^\circ\sin\alpha_{i-1} + 2|e_{i-1}||e|\tan 89.5^\circ\sin\alpha_{i-1} + x^2 + |e|^2\cos^2\alpha_{i-1} - 2|e|x\cos\alpha_{i-1} = m^2$. Simplifying the previous equation we get $(\tan^2 89.5^\circ + 1)x^2 - 2(|e_{i-1}|\tan^2 89.5^\circ + |e|\tan 89.5^\circ\sin\alpha_{i-1} + |e|\cos\alpha_{i-1})x + |e_{i-1}|^2\tan^2 89.5^\circ + 2|e_{i-1}||e|\tan 89.5^\circ\sin\alpha_{i-1} + |e|^2 - m^2 = 0$. Thus $l(\overline{\alpha_i})$ intersects $k(v,m)$ twice if and only if $(|e_{i-1}|\tan^2 89.5^\circ+|e|\tan 89.5^\circ\sin\alpha_{i-1}+|e|\cos\alpha_{i-1})^2-(\tan^2 89.5^\circ+1)(|e_{i-1}|^2\tan^2 89.5^\circ+2|e_{i-1}||e|\tan 89.5^\circ\sin\alpha_{i-1}+|e|^2-m^2)\geq 0$. To prove that the last inequality holds, we distinguish two cases, namely the one in which $|e|\geq|e_{i-1}|$ and the one in which $|e_{i-1}|\geq|e|$.

First, suppose that $|e|\geq|e_{i-1}|$, that is, $m=|e|$. Then, we have to prove that $(|e_{i-1}|\tan^2 89.5^\circ+|e|\tan 89.5^\circ\sin\alpha_{i-1}+|e|\cos\alpha_{i-1})^2-$
$(\tan^2 89.5^\circ+1)(|e_{i-1}|^2\tan^2 89.5^\circ+2|e_{i-1}||e|\tan 89.5^\circ\sin\alpha_{i-1})\geq 0$, that is,
$|e_{i-1}|^2\tan^4 89.5^\circ+|e|^2\tan^2 89.5^\circ\sin^2\alpha_{i-1}+|e|^2\cos^2\alpha_{i-1}+$
$2|e_{i-1}||e|\tan^3 89.5^\circ\sin\alpha_{i-1}+2|e_{i-1}||e|\tan^2 89.5^\circ\cos\alpha_{i-1}+$
$2|e|^2\tan 89.5^\circ\sin\alpha_{i-1}\cos\alpha_{i-1}-|e_{i-1}|^2\tan^4 89.5^\circ-|e_{i-1}|^2\tan^2 89.5^\circ-$
$2|e_{i-1}||e|\tan^3 89.5^\circ\sin\alpha_{i-1}-2|e_{i-1}||e|\tan 89.5^\circ\sin\alpha_{i-1}\geq 0$. Simplifying the previous one and using $|e|\geq|e_{i-1}|$ and $2|e|\cos\alpha_{i-1}\leq|e_{i-1}|$ (by Lemma 5), we get that, in order to prove the previous inequality, it suffices to prove that
$|e_{i-1}|^2\tan^2 89.5^\circ\sin^2\alpha_{i-1}+|e_{i-1}|^2\cos^2\alpha_{i-1}+2|e_{i-1}|^2\tan^2 89.5^\circ\cos\alpha_{i-1}+$
$2|e_{i-1}|^2\tan 89.5^\circ\sin\alpha_{i-1}\cos\alpha_{i-1}-|e_{i-1}|^2\tan^2 89.5^\circ-$
$4|e_{i-1}|^2\tan 89.5^\circ\sin\alpha_{i-1}\cos\alpha_{i-1}\geq 0$. Moreover, since $\sin 60^\circ\leq\sin\alpha_{i-1}\leq\sin 61^\circ$ and $\cos 61^\circ\leq\cos\alpha_{i-1}\leq\cos 60^\circ$ (by hypothesis and by Lemma 2), we get that the previous inequality is implied by $|e_{i-1}|^2(\tan^2 89.5^\circ\sin^2 60^\circ+\cos^2 61^\circ+$
$2\tan^2 89.5^\circ\cos 61^\circ+2\tan 89.5^\circ\sin 60^\circ\cos 61^\circ-\tan^2 89.5^\circ-$
$4\tan 89.5^\circ\sin 61^\circ\cos 60^\circ)>9345|e_{i-1}|^2>0$. Thus, if $|e|\geq|e_{i-1}|$ then $l(\overline{\alpha_i})$ intersects $k(v,m)$ twice.

Second, suppose that $|e_{i-1}|\geq|e|$, that is, $m=|e_{i-1}|$. Then, we have to prove that $(|e_{i-1}|\tan^2 89.5^\circ+|e|\tan 89.5^\circ\sin\alpha_{i-1}+|e|\cos\alpha_{i-1})^2-$
$(\tan^2 89.5^\circ+1)(|e_{i-1}|^2\tan^2 89.5^\circ+2|e_{i-1}||e|\tan 89.5^\circ\sin\alpha_{i-1}+|e|^2-|e_{i-1}|^2)\geq 0$, that is,
$|e_{i-1}|^2\tan^4 89.5^\circ+|e|^2\tan^2 89.5^\circ\sin^2\alpha_{i-1}+|e|^2\cos^2\alpha_{i-1}+$
$2|e_{i-1}||e|\tan^3 89.5^\circ\sin\alpha_{i-1}+2|e_{i-1}||e|\tan^2 89.5^\circ\cos\alpha_{i-1}+$
$2|e|^2\tan 89.5^\circ\sin\alpha_{i-1}\cos\alpha_{i-1}-|e_{i-1}|^2\tan^4 89.5^\circ-|e_{i-1}|^2\tan^2 89.5^\circ-$
$2|e_{i-1}||e|\tan^3 89.5^\circ\sin\alpha_{i-1}-2|e_{i-1}||e|\tan 89.5^\circ\sin\alpha_{i-1}-|e|^2\tan^2 89.5^\circ-|e|^2+|e_{i-1}|^2\tan^2 89.5^\circ+$
$|e_{i-1}|^2\geq 0$. Simplifying the previous one and using $|e|\leq|e_1|$ and $|e|\geq 2|e_1|\cos\alpha_{i-1}$ (by Lemma 5), we get that, in order to prove the previous inequality, it suffices to prove that $4|e_{i-1}|^2\tan^2 89.5^\circ\sin^2\alpha_{i-1}\cos^2\alpha_{i-1}+4|e_{i-1}|^2\cos^4\alpha_{i-1}+4|e_{i-1}|^2\tan^2 89.5^\circ\cos^2\alpha_{i-1}+$
$8|e_{i-1}|^2\tan 89.5^\circ\sin\alpha_{i-1}\cos^3\alpha_{i-1}-|e_{i-1}|^2\tan^2 89.5^\circ-2|e_{i-1}|^2\tan 89.5^\circ\sin\alpha_{i-1}-|e_{i-1}|^2\tan^2 89.5^\circ-$
$|e_{i-1}|^2+|e_{i-1}|^2\tan^2 89.5^\circ+|e_{i-1}|^2\geq 0$. Moreover, since $\sin 60^\circ\leq\sin\alpha_{i-1}\leq\sin 61^\circ$ and $\cos 61^\circ\leq\cos\alpha_{i-1}\leq\cos 60^\circ$ (by hypothesis and by Lemma 2), we get that the previous inequality is implied by $|e_{i-1}|^2(4\tan^2 89.5^\circ\sin^2 60^\circ\cos^2 61^\circ+4\cos^4 61^\circ+4\tan^2 89.5^\circ\cos^2 61^\circ+$
$8\tan 89.5^\circ\sin 60^\circ\cos^3 61^\circ-\tan^2 89.5^\circ-2\tan 89.5^\circ\sin 61^\circ)\geq 8363|e_{i-1}|^2>0$. Thus, even if $|e_{i-1}|\geq|e|$ then $l(\overline{\alpha_i})$ intersects $k(v,m)$ twice. □

**Lemma 12** *The distance between $u_i$ and $h_{\overline{\alpha_i}}$ is at least $1.604|e_{i-1}|$.*

**Proof:** By the proof of Lemma 11, we have that the intersection points of $k(v,m)$ with $l(\overline{\alpha_i})$ satisfy $(\tan^2 89.5^\circ+1)x^2-2(|e_{i-1}|\tan^2 89.5^\circ+|e|\tan 89.5^\circ\sin\alpha_{i-1}+|e|\cos\alpha_{i-1})x+|e_{i-1}|^2\tan^2 89.5^\circ+2|e_{i-1}||e|\tan 89.5^\circ\sin\alpha_{i-1}+|e|^2-m^2=0$. To lower bound the distance between $u_i$ and $h_{\overline{\alpha_i}}$ we distinguish two cases, namely the one in which $|e|\geq|e_{i-1}|$ and the one in which $|e_{i-1}|\geq|e|$.

First, suppose that $|e|\geq|e_{i-1}|$. By the computation in the proof of Lemma 11, the discriminant of the equation describing the $x$-coordinates of the intersections of $k(v,m)$ with $l(\overline{\alpha_i})$ is at least $9345|e_{i-1}|^2$. Hence, since $\sin 60^\circ\leq\sin\alpha_{i-1}\leq\sin 61^\circ$ and

$\cos 61^\circ \leq \cos\alpha_{i-1} \leq \cos 60^\circ$ (by hypothesis and by Lemma 2) and since $|e| \geq |e_{i-1}|$ and $2|e|\cos\alpha_{i-1} \leq |e_{i-1}|$ (by Lemma 5), we get that $h_{\overline{\alpha_i}}$ has $x$-coordinate which is at least $\frac{|e_{i-1}|\tan^2 89.5^\circ + |e_{i-1}|\tan 89.5^\circ \sin 60^\circ + |e_{i-1}|\cos 61^\circ + |e_{i-1}|\sqrt{9345}}{\tan^2 89.5^\circ + 1} > 1.014|e_{i-1}|$. Plugging such a lower bound into the equation $y = \tan 89.5^\circ(x - |e_{i-1}|)$ of $l(\overline{\alpha_i})$ we get that the $y$-coordinate of $h_{\overline{\alpha_i}}$ is at least $1.604|e_{i-1}|$. Hence, the distance between $h_{\overline{\alpha_i}}$ and $u_i$ is at least $|e_{i-1}|\sqrt{(1.604)^2 + (0.014)^2} > 1.604|e_{i-1}|$.

Second, suppose that $|e_{i-1}| \geq |e|$. By the computation in the proof of Lemma 11, the discriminant of the equation describing the $x$-coordinates of the intersections of $k(v,m)$ with $l(\overline{\alpha_i})$ is at least $8363|e_{i-1}|^2$. Hence, since $\sin 60^\circ \leq \sin\alpha_{i-1} \leq \sin 61^\circ$ and $\cos 61^\circ \leq \cos\alpha_{i-1} \leq \cos 60^\circ$ (by hypothesis and by Lemma 2) and since $|e| \leq |e_{i-1}|$ and $|e| \geq 2|e_{i-1}|\cos\alpha_{i-1}$ (by Lemma 5), we get that $h_{\overline{\alpha_i}}$ has $x$-coordinate which is at least $\frac{|e_{i-1}|\tan^2 89.5^\circ + 2|e_{i-1}|\tan 89.5^\circ \sin 60^\circ \cos 61^\circ + 2|e_{i-1}|\cos^2 61^\circ + |e_{i-1}|\sqrt{8363}}{\tan^2 89.5^\circ + 1} > 1.014|e_{i-1}|$. Again, this yields a $1.604|e_{i-1}|$ lower bound for the $y$-coordinate of $h_{\overline{\alpha_i}}$ and to a $1.604|e_{i-1}|$ lower bound for the the distance between $h_{\overline{\alpha_i}}$ and $u_i$. □

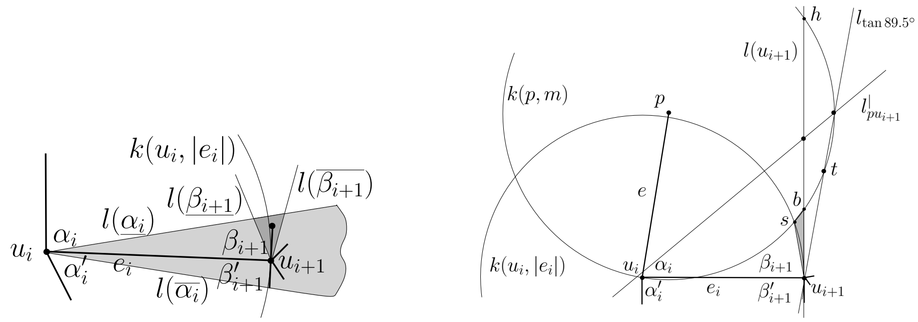
We are now ready to state the following:

**Lemma 13** *Suppose that $\alpha_{i-1} \leq 61^\circ$, that $\beta'_i, \beta'_{i+1} \geq 89.5^\circ$, and that $|e_i| \leq \frac{|e_{i-1}|}{10}$. Then, $Cl(u_i)$ is inside a bounded region $R_i$ that is a subset of $W_i$.*

**Proof:** Let $R_i$ be the bounded region delimited by $l(\underline{\alpha_i})$ from the left, by $l(\overline{\alpha_i})$ from the right, and by $k(v,m)$ from above. We prove that $Cl(u_i)$ is inside $R_i$.

First, we prove that $u_{i+1}$ is in $R_i$. By the assumption that $\alpha_{i-1} \leq 61^\circ$ and by Lemma 6, $u_{i+1}$ is not to the left of $l(\underline{\alpha_i})$. By the assumption that $\beta'_i \geq 89.5^\circ$ and by Lemma 10, $u_{i+1}$ is not to the right of $l(\overline{\alpha_i})$. Hence, $u_{i+1}$ is in $W_i$. By the assumption that $\alpha_{i-1} \leq 61^\circ$ and by Lemma 11, $k(v,m)$ intersects $l(\overline{\alpha_i})$. Moreover, $v$ is to the left of $l(\underline{\alpha_i})$. Namely, $v \equiv (|e|\cos\alpha_{i-1}, |e|\sin\alpha_{i-1})$. Further, if $y = |e|\sin\alpha_{i-1}$, then the $x$-coordinate of $l(\underline{\alpha_i})$ is $x = |e_{i-1}| - (|e|\sin\alpha_{i-1})/\tan 89.5^\circ$. Since $|e_{i-1}| \geq 2|e|\cos\alpha_{i-1}$ (by Lemma 5) and $60^\circ \leq \alpha_{i-1} \leq 61^\circ$ (by assumption and by Lemma 2), we have $|e_{i-1}| - |e|\sin\alpha_{i-1}/\tan 89.5^\circ \geq 2\cos 61^\circ|e| - |e|\sin 61^\circ/\tan 89.5^\circ \geq 0.96|e| > |e|\cos 60^\circ \geq |e|\cos\alpha_{i-1}$. Since $v$ is to the left of $l(\underline{\alpha_i})$ and since $k(v,m)$ intersects $l(\overline{\alpha_i})$, there exists a bounded region $F_i$ of $W_i$, delimited by $k(v,m)$ from above and from below, by $l(\underline{\alpha_i})$ from the left, and by $l(\overline{\alpha_i})$ from the right, in which $u_{i+1}$ can not lie, as otherwise Lemma 1 would be violated. By Lemma 12, the distance between $u_i$ and every point above $F_i$ is at least $1.604|e_{i-1}|\cos 0.5^\circ > 1.4|e_{i-1}|$. Hence, by the assumption that $|e_i| \leq |e_{i-1}|/10$, $u_{i+1}$ is not above $F_i$. It follows that $u_{i+1}$ is in $R_i$.

Next, we prove that $w$ is in $R_i$. Observe that $\beta_{i+1} \leq 90.5^\circ$, by the assumption that $\beta'_{i+1} \geq 89.5^\circ$ and since the three angles incident to $u_{i+1}$ and different from $\beta_{i+1}$ and $\beta'_{i+1}$ sum up to at least $180^\circ$ (by Lemma 2). Hence, $e^*$ can not cross $l(\overline{\alpha_i})$. Since $\beta_i, \beta_{i+1} \leq 90.5^\circ$, the angle defined by a clockwise rotation bringing a horizontal line to coincide with $e^*$ is at most $1^\circ$. Since the $x$-coordinate of $u_{i+1}$ is at most $|e_{i-1}| + \frac{|e_{i-1}|\sin 0.5}{10}$, the $y$-coordinate of the line through $e^*$ if $x = |e|\cos\alpha_{i-1}$ is at most $\frac{|e_{i-1}|}{10} + \tan 1^\circ(|e_{i-1}| + \frac{|e_{i-1}|\sin 0.5}{10} - |e|\cos\alpha_{i-1}) \leq \frac{|e|}{20\cos 61^\circ} + \tan 1^\circ(\frac{|e|}{2\cos 61^\circ} + \frac{|e|\sin 0.5}{20\cos 61^\circ} - |e|\cos 61^\circ) < 0.112|e| < |e|\sin\alpha_{i-1}$, since $\alpha_{i-1} \leq 61^\circ$, by assumption, and $2|e_{i-1}|\cos\alpha_{i-1} \leq |e|$, by Lemma 5. Then, the line through $e^*$ crosses the vertical line through $v$ below $v$. Since the $y$-coordinate of every point above $F_i$ is at least $1.4|e_{i-1}|$, by Lemma 12, $e^*$ can not cross $k(v,m)$. Further, the region $S_i$ bounded by $e$ from the left, by $e_{i-1}$ from below, by $l(\underline{\alpha_i})$ from the right, and by the horizontal line




(a) (b)

Figure 7: (a) Illustration for Lemma 14. The dark-shaded region is $R_{i+1}$. (b) Illustration for Lemma 15. The dark-shaded region is $R_1$. To improve the readability, angles and edge lengths in the illustrations do not correspond to actual angles and edge lengths.

through $v$ from above entirely belongs to $k(v, m) \cup k(u_{i-1}, |e_{i-1}|)$, by Lemma 9; since the $y$-coordinate of $w$ is at most $0.112|e| < |e| \sin \alpha_{i-1}$, if $e^*$ crosses $l(\underline{\alpha_i})$, then either $w$ is in $S_i$, thus violating Lemma 1, or $e^*$ crosses an edge of $T^*$, thus violating Lemma 4. Hence, $w$ is in $R_i$.

Finally, consider the rest of $Cl(u_i)$. The angle defined by a clockwise rotation bringing an edge $g_1$ of $Cl(u_i)$ to overlap with the next edge $g_2$ of $Cl(u_i)$ is at most $120^\circ$, since the four other angles incident to the vertex shared by $g_1$ and $g_2$ sum up to at least $240^\circ$ (by Lemma 2). Hence, no edge of $Cl(u_i)$ crosses $l(\overline{\alpha_i})$ or $k(v, m)$, as otherwise such an edge crosses an edge of $T^*$, thus violating Lemma 4. Moreover, no edge of $Cl(u_i)$ crosses $l(\underline{\alpha_i})$, as otherwise either one end-vertex of such an edge is in $S_i$, thus violating Lemma 1, or the edge crosses an edge of $T^*$, thus violating Lemma 4. □

Lemma 13 assumes that $|e_i| \leq \frac{|e_{i-1}|}{10}$. The reason why we can assume such a ratio will be made clear at the end of the section.

We now prove that the condition that the clockwise path of each vertex is inside a bounded region propagates along the vertices of the backbone. Refer to Fig. 7(a).

**Lemma 14** *Suppose that $\alpha_i \geq 89.5^\circ$, that $\beta'_{i+1} \geq 89.5^\circ$, and that $Cl(u_i)$ is in a bounded region $R_i$ that is a subset of a wedge $W_i$ centered at $u_i$ with angle $1^\circ$. Then, $\beta_{i+1} \geq 89.5^\circ$. Moreover, $Cl(u_{i+1})$ is in a bounded region $R_{i+1}$ that is a subset of a wedge $W_{i+1}$ centered at $u_{i+1}$ with angle $1^\circ$.*

**Proof:** Since $Cl(u_i)$ is in $R_i$, it follows that $u_{i+1}$ is in $R_i$. Then, $w$ is not inside $k(u_i, |e_i|)$, as otherwise Lemma 1 would be violated. Hence, the minimum value of $\widehat{u_i u_{i+1} w} = \beta_{i+1}$ is achieved if $w$ is on $k(u_i, |e_i|)$, inside $R_i$, and hence inside $W_i$. If $w$ is on $k(u_i, |e_i|)$, then triangle $\Delta(u_i u_{i+1} w)$ is isosceles. Since $\widehat{u_{i+1} u_i w} \leq 1^\circ$, then $\beta_{i+1} \geq 89.5$, thus proving the first part of the lemma.

Next, let $l(\underline{\beta_{i+1}})$ ($l(\overline{\beta_{i+1}})$) be the half-line starting at $u_{i+1}$ such that a $89.5^\circ$ (resp. $90.5^\circ$) clockwise rotation around $u_{i+1}$ brings $e_i$ to overlap with $l(\underline{\beta_{i+1}})$ (resp. with $l(\overline{\beta_{i+1}})$). Define $R_{i+1}$ as the intersection of $R_i$ and the wedge delimited by $l(\underline{\beta_{i+1}})$ and $l(\overline{\beta_{i+1}})$. Then $R_{i+1}$ is bounded as $R_i$ is; further, $R_{i+1}$ is a subset of a wedge $W_{i+1}$ centered at $u_{i+1}$ with angle $1^\circ$. We prove that $Cl(u_{i+1})$ lies inside $R_{i+1}$.

Since $\beta'_{i+1} \geq 89.5^\circ$ and the three angles incident to $u_{i+1}$ and different from $\beta_{i+1}$ and $\beta'_{i+1}$ sum up to at least $180^\circ$, it holds $\beta_{i+1} \leq 90.5^\circ$. Since $Cl(u_i)$ is in $R_i$ and the angle defined by a clockwise rotation bringing an edge $g_1$ of $Cl(u_i)$ to overlap with the next edge $g_2$ of $Cl(u_i)$ is at most $120^\circ$, as the four other angles incident to the vertex shared by $g_1$ and $g_2$ sum up to at least $240^\circ$ (by Lemma 2), then every vertex of $Cl(u_{i+1})$ is not to the right of $l(\overline{\beta_{i+1}})$, as otherwise an edge of such a path crosses $e_i$ or $(u_{i+1}, w)$, thus contradicting Lemma 4. The region delimited by $e_i$ from below, by $l(\underline{\beta_{i+1}})$ from the right, and by $l(\underline{\alpha_i})$ from above is a subset of $k(u_i, |e_i|)$ since the line through $u_{i+1}$ and through the intersection point of $k(u_i, |e_i|)$ and $l(\underline{\alpha_i})$ forms with $e_i$ an angle which is at least $89.5^\circ$. Hence, if an edge of $Cl(u_{i+1})$ crosses $l(\overline{\beta_{i+1}})$, then either a vertex of $Cl(u_{i+1})$ is in $k(u_i, |e_i|)$, thus violating Lemma 1, or an edge of $Cl(u_{i+1})$ crosses $e_i$ or $(u_{i+1}, w)$, thus violating Lemma 4. It follows that $Cl(u_{i+1})$ is in $R_{i+1}$. □

We now deal with the edge lengths in any MST embedding of $T^*$. Consider a backbone edge $e_i$=$(u_i, u_{i+1})$ such that the outgoing angle $\alpha_i$ is small. Let $e^* = (u_{i+1}, q)$ ($e = (u_i, p)$) be the edge following $e_i$ in the clockwise (resp. counterclockwise) order of the edges incident to $u_{i+1}$ (resp. to $u_i$). Let $\alpha_i$ and $\beta_{i+1}$ be the angles delimited by $e_i$ and $e$ and by $e_i$ and $e^*$, respectively. See Fig. 7(b). The following lemma asserts that is $\alpha_i$ is small and $\beta_{i+1}$ is large then $e^*$ is much shorter than $e_i$.

**Lemma 15** *Suppose that $\alpha_i \leq 61^\circ$ and $\beta_{i+1} \leq 90.5^\circ$. Then, it holds $\frac{|e^*|}{|e_i|} \leq 0.073$.*

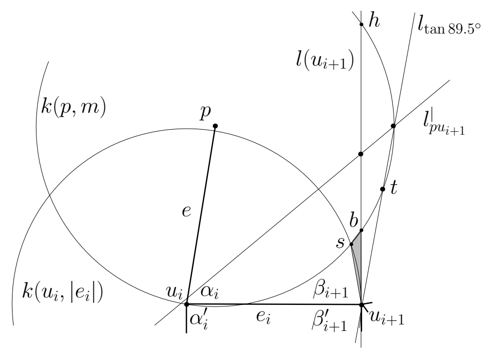



Figure 8: Illustration for Lemma 15. The dark-shaded region is $R_1$. To improve the readability, angles and edge lengths in the illustrations do not correspond to actual angles and edge lengths.

**Proof:** Assume w.l.o.g. up to a rotation, a reflection, and a translation of the drawing, that $e_i$ is horizontal with $u_{i+1}$ to the right of $u_i$. Assume that $u_i$ has coordinates $(0, 0)$. Let $m = \max\{|e|, |e_i|\}$. Further, let $l(u_{i+1})$ be the vertical line through $u_{i+1}$ and $l^{|}_{pu_{i+1}}$ the line orthogonal to $\overline{pu_{i+1}}$ through the midpoint of such a segment. Let $b$ and $h$ be the lowest and the highest intersection point of $k(p, m)$ and $l(u_{i+1})$, respectively. Let $s$ be the rightmost intersection point of $k(p, m)$ and $k(u_i, |e_i|)$. Refer to Fig. 8.

We distinguish two cases, namely the one in which $\beta_{i+1} \leq 90^\circ$ and the one in which $90^\circ < \beta_{i+1} \leq 90.5^\circ$. By assumption, no other values of $\beta_{i+1}$ have to be considered to prove the lemma.

Suppose that $\beta_{i+1} \leq 90^\circ$. We claim that the maximum value of $|e^*|$ is achieved when $q$ is either at $b$ or at $s$. Namely, by Lemma 1, we have that: (i) $q$ is outside $k(p, m)$; (ii) $q$ is in the half-plane that is delimited by $l^{|}_{pu_{i+1}}$ and that does not contain $p$; and (iii) $q$ is outside $k(u_i, |e_i|)$. Further, $q$ is not to the right of $l(u_{i+1})$ since $\beta_{i+1} \leq 90^\circ$. Hence, as long as $l^{|}_{pu_{i+1}}$ intersects $l(u_{i+1})$ below $h$, $q$ is in the region $R_1$ bounded by $l(u_{i+1})$ from the right, by $k(p,m)$ from above, and by $k(u_i,|e_i|)$ from below. Such a region is a subset of triangle $\Delta(u_{i+1},s,b)$, since $\overline{sb}$ is a chord of $k(p,m)$ and $\overline{u_{i+1}s}$ is a chord of $k(u_i,|e_i|)$. Hence, the farthest point from $u_{i+1}$ inside $R_1$ is either $b$ or $s$.

**Claim 1** *The intersection of $l^{|}_{pu_{i+1}}$ and $l(u_{i+1})$ is below $h$.*

**Proof:** Line $l^{|}_{pu_{i+1}}$ has equation $y - \frac{|e|\sin\alpha_i}{2} = \frac{|e_i|-|e|\cos\alpha_i}{|e|\sin\alpha_i}(x - \frac{|e|\cos\alpha_i+|e_i|}{2})$. Intersecting such a line with $l(u_{i+1})$, that has equation $x = |e_i|$, we get $y = \frac{|e|\sin\alpha_i}{2} + \frac{|e_i|^2}{|e|\sin\alpha_i} - \frac{|e_i||e|\cos\alpha_i}{|e|\sin\alpha_i} - \frac{|e_i||e|\cos\alpha_i}{2|e|\sin\alpha_i} - \frac{|e_i|^2}{2|e|\sin\alpha_i} + \frac{|e|^2\cos^2\alpha_i}{2|e|\sin\alpha_i} + \frac{|e_i||e|\cos\alpha_i}{2|e|\sin\alpha_i}$. Simplifying the previous formula, the $y$-coordinate of the intersection of $l^{|}_{pu_{i+1}}$ with $l(u_{i+1})$ is $y = \frac{|e|^2+|e_i|^2-2|e_i||e|\cos\alpha_i}{2|e|\sin\alpha_i}$.

Next, we compute the intersection of $k(p,m)$ with $l(u_{i+1})$. The equation of $k(p,m)$ is $(x-(|e|\cos\alpha_i))^2 + (y-(|e|\sin\alpha_i))^2 = m^2$. Intersecting such a curve with $x = |e_i|$ we get $y^2 - 2|e|y\sin\alpha_i + |e|^2 + |e_i|^2 - 2|e_i||e|\cos\alpha_i = m^2$, that is, the $y$-coordinate of $h$ is $y = |e|\sin\alpha_i + \sqrt{|e|^2\sin^2\alpha_i - |e|^2 - |e_i|^2 + m^2 + 2|e_i||e|\cos\alpha_i}$.

Suppose that $|e| \geq |e_i|$, that is, $m = |e|$. Then, in order to prove that $l^{|}_{pu_{i+1}}$ intersects $l(u_{i+1})$ below $h$, we have to show that $\frac{|e|^2+|e_i|^2-2|e_i||e|\cos\alpha_i}{2|e|\sin\alpha_i} <$
$|e|\sin\alpha_i + \sqrt{|e|^2\sin^2\alpha_i - |e_i|^2 + 2|e_i||e|\cos\alpha_i}$.

Since $|e| \geq |e_i|$ and $2|e|\cos\alpha_i \leq |e_i|$ (by Lemma 5) and since $\sin\alpha_i \geq \sin 60^{\circ}$ and $\cos\alpha_i \geq \cos 61^{\circ}$ (by hypothesis and by Lemma 2), we get $\frac{|e|^2+|e_i|^2-2|e_i||e|\cos\alpha_i}{2|e|\sin\alpha_i} \leq \frac{|e|^2+|e^2|-4|e|^2\cos^2\alpha_i}{2|e|\sin\alpha_i} = \frac{|e|-2|e|\cos^2\alpha_i}{\sin\alpha_i} \leq |e|\frac{1-2\cos^2 61^{\circ}}{\sin 60^{\circ}} < 0.61189|e|$.

On the other hand, $|e|\sin\alpha_i + \sqrt{|e|^2\sin^2\alpha_i - |e_i|^2 + 2|e_i||e|\cos\alpha_i} \geq$
$|e|\sin\alpha_i + \sqrt{|e|^2\sin^2\alpha_i - |e|^2 + 4|e|^2\cos^2\alpha_i} \geq |e|\sin 60^{\circ} + \sqrt{|e|^2\sin^2 60^{\circ} - |e|^2 + 4|e|^2\cos^2 61^{\circ}} =$
$|e|(\sin 60^{\circ} + \sqrt{\sin^2 60^{\circ} - 1 + 4\cos^2 61^{\circ}} > 1.6967|e|$. Thus, if $|e| \geq |e_i|$ then $l^{|}_{pu_{i+1}}$ intersects $l(u_{i+1})$ below $h$.

Next, suppose that $|e_i| \geq |e|$, that is, $m = |e_i|$. Then, in order to prove that $l^{|}_{pu_{i+1}}$ intersects $l(u_{i+1})$ below $h$, we have to show that $\frac{|e|^2+|e_i|^2-2|e_i||e|\cos\alpha_i}{2|e|\sin\alpha_i} < |e|\sin\alpha_i + \sqrt{|e|^2\sin^2\alpha_i - |e|^2 + 2|e_i||e|\cos\alpha_i}$.

Since $|e_i| \geq |e|$ and $2|e_i|\cos\alpha_i \leq |e|$ (by Lemma 5) and since $\sin\alpha_i \geq \sin 60^{\circ}$ and $\cos\alpha_i \geq \cos 61^{\circ}$ (by hypothesis and by Lemma 2), we get $\frac{|e|^2+|e_i|^2-2|e_i||e|\cos\alpha_i}{2|e|\sin\alpha_i} \leq \frac{|e_i|^2+|e_i|^2-4|e_i|^2\cos^2\alpha_i}{4|e_i|\cos\alpha_i\sin\alpha_i} = \frac{|e_i|-2|e_i|\cos^2\alpha_i}{2\cos\alpha_i\sin\alpha_i} \leq |e_i|\frac{1-2\cos^2 61^{\circ}}{2\cos 61^{\circ}\sin 60^{\circ}} < 0.6311|e_i|$.

On the other hand, $|e|\sin\alpha_i + \sqrt{|e|^2\sin^2\alpha_i - |e|^2 + 2|e_i||e|\cos\alpha_i} \geq$
$2|e_i|\sin\alpha_i\cos\alpha_i + \sqrt{4|e_i|^2\cos^2\alpha_i\sin^2\alpha_i - |e_i|^2 + 4|e_i|^2\cos^2\alpha_i} \geq$
$2|e_i|\sin 60^{\circ}\cos 61^{\circ} + \sqrt{4|e_i|^2\sin^2 60^{\circ}\cos^2 61^{\circ} - |e_i|^2 + 4|e_i|^2\cos^2 61^{\circ}} =$
$|e_i|(2\sin 60^{\circ}\cos 61^{\circ} + \sqrt{4\sin^2 60^{\circ}\cos^2 61^{\circ} - 1 + 4\cos^2 61^{\circ}}) > 1.643|e_i|$. Thus, even if $|e_i| \geq |e|$ then $l^{|}_{pu_{i+1}}$ intersects $l(u_{i+1})$ below $h$. □

We now distinguish the two cases in which $|e^*| = |\overline{u_{i+1}b}|$ and $|e^*| = |\overline{u_{i+1}s}|$.

*Suppose that the farthest point from $u_{i+1}$ inside $R_1$ is $b$.* We compute $|\overline{u_{i+1}b}|$. The equation of $k(p,m)$ is $(x-|e|\cos\alpha_i)^2 + (y-|e|\sin\alpha_i)^2 = m^2$. Setting $x = |e_i|$ into such an equation we get the $y$-coordinate of $b$, that is
$y = |e|\sin\alpha_i - \sqrt{m^2 - |e_i|^2 + 2|e_i||e|\cos\alpha_i - |e|^2\cos^2\alpha_i} = |\overline{u_{i+1}b}|$.

First, suppose that $|e_i| \geq |e|$. Then, $|\overline{u_{i+1}b}| = |e|\sin\alpha_i - \sqrt{2|e_i||e|\cos\alpha_i - |e|^2\cos^2\alpha_i} \leq$
$|e|\sin\alpha_i - \sqrt{2|e|^2\cos\alpha_i - |e|^2\cos^2\alpha_i} \leq |e_i|(\sin\alpha_i - \sqrt{2\cos\alpha_i - \cos^2\alpha_i})$. Studying the

derivative of $2\cos\alpha_i - \cos^2\alpha_i$, we get that such a function is monotonically decreasing with $\alpha_i$, hence $|\overline{u_{i+1}b}| \leq |e_i|(\sin 61^\circ - \sqrt{2\cos 61^\circ - \cos^2 61^\circ}) < 0.0176$. Second, suppose that $|e| \geq |e_i|$. Then, $|\overline{u_{i+1}b}| = |e|\sin\alpha_i - \sqrt{|e|^2 - |e_i|^2 + 2|e_i||e|\cos\alpha_i - |e|^2\cos^2\alpha_i} \leq |e|\sin\alpha_i - \sqrt{2|e_i||e|\cos\alpha_i - |e|^2\cos^2\alpha_i} \leq |e|\sin\alpha_i - \sqrt{3|e|^2\cos^2\alpha_i} = |e|(\sin\alpha_i - \sqrt{3}\cos\alpha_i) \leq |e_i|\frac{\sin\alpha_i - \sqrt{3}\cos\alpha_i}{2\cos\alpha_i}$, where we used twice $|e_i| \geq 2|e|\cos\alpha_i$, which holds by Lemma 5. Since $\tan\alpha_i$ is monotonically increasing with $\alpha_i$ between $60^\circ$ and $61^\circ$, we get $\frac{|\overline{u_{i+1}b}|}{|e_i|} \leq \frac{\tan 61^\circ}{2} - \frac{\sqrt{3}}{2} = 0.036$.

*Suppose that the farthest point from $u_{i+1}$ inside $R_1$ is $s$.* We have that $s \equiv (|e_i|\cos\gamma, |e_i|\sin\gamma)$, where $\gamma = \widehat{u_{i+1}u_i s}$. Observe that, by Lemma 6, $\gamma = \frac{180^\circ - \beta_i}{2} \leq \alpha_i - 60^\circ \leq 1^\circ$. Hence, $|\overline{u_{i+1}s}| = \sqrt{(|e_i|\sin\gamma)^2 + (|e_i| - |e_i|\cos\gamma)^2} = |e_i|\sqrt{2 - 2\cos\gamma} \leq |e_i|\sqrt{2 - 2\cos 1^\circ} < 0.0175|e_i|$, where we used the fact that $\cos\gamma$ is monotonically decreasing between $0^\circ$ and $1^\circ$.

Suppose that $90^\circ < \beta_{i+1} \leq 90.5^\circ$. We claim that $|e^*|$ is at most $|u_{i+1}t|$, where $t$ is the intersection point of $k(p, m)$ and the line $l_{\tan 89.5^\circ}$ through $u_{i+1}$ with slope $\tan 89.5^\circ$. First, $p$ is to the left of $l(u_{i+1})$, since $|e|\cos\alpha_i < 2|e|\cos\alpha_i \leq |e_i|$, which holds by Lemma 5; further, by Lemma 11 (where $\alpha_i$, $k(p,m)$, and $l_{\tan 89.5^\circ}$ replace $\alpha_{i-1}$, $k(v,m)$, and $l(\overline{\alpha_i})$, resp.), $l_{\tan 89.5^\circ}$ intersects $k(p,m)$ twice. Denote by $l^{|}_{pu_{i+1}}$ the line orthogonal to $\overline{pu_{i+1}}$ through the midpoint of $\overline{pu_{i+1}}$. We have the following:

**Claim 2** *The distance between $u_{i+1}$ and the intersection point $h^{|}(p, u_{i+1}, \tan 89.5^\circ)$ of $l^{|}_{pu_{i+1}}$ and $l_{\tan 89.5^\circ}$ is at most $0.66|e_{i-1}|$.*

**Proof:** First, we derive the equation of $l^{|}_{pu_{i+1}}$. Such a line passes through the midpoint of $\overline{pu_{i+1}}$, that has coordinates $(\frac{|e|\cos\alpha_i + |e_i|}{2}, \frac{|e|\sin\alpha_i}{2})$. Moreover, $l^{|}_{pu_{i+1}}$ is orthogonal to the line through $v$ and $u_{i+1}$, that has equation $y = \frac{x|e|\sin\alpha_i - |e_i||e|\sin\alpha_i}{|e|\cos\alpha_i - |e_i|}$. Hence, the slope of $l^{|}_{pu_{i+1}}$ is $\frac{|e_i| - |e|\cos\alpha_i}{|e|\sin\alpha_i}$. Then, $l^{|}_{pu_{i+1}}$ has equation $y - \frac{|e|\sin\alpha_i}{2} = \frac{|e_i| - |e|\cos\alpha_i}{|e|\sin\alpha_i}(x - \frac{|e|\cos\alpha_i + |e_i|}{2})$. Second, the equation of $l(\overline{\alpha_i})$ is $y = \tan 89.5^\circ(x - |e_i|)$. Intersecting such two lines we get $\tan 89.5^\circ(x - |e_i|) = \frac{|e|\sin\alpha_i}{2} + \frac{|e_i| - |e|\cos\alpha_i}{|e|\sin\alpha_i}(x - \frac{|e|\cos\alpha_i + |e_i|}{2})$, that is $x = \frac{\tan 89.5^\circ|e_i| + \frac{|e|\sin\alpha_i}{2} + \frac{(|e_i| - |e|\cos\alpha_i)(-|e|\cos\alpha_i - |e_i|)}{2|e|\sin\alpha_i}}{\tan 89.5^\circ + \frac{|e|\cos\alpha_i - |e_i|}{|e|\sin\alpha_i}} = \frac{\tan 89.5^\circ|e_i| + \frac{|e|\sin\alpha_i}{2} + \frac{|e|^2\cos^2\alpha_i - |e_i|^2}{2|e|\sin\alpha_i}}{\tan 89.5^\circ + \frac{|e|\cos\alpha_i - |e_i|}{|e|\sin\alpha_i}}$.

Suppose that $|e| \geq |e_i|$. Then, by Lemma 5, $e \leq \frac{|e_i|}{2\cos\alpha_i}$. Using the last two inequalities we get $x \leq \frac{\tan 89.5^\circ|e_i| + \frac{|e_i|\sin\alpha_i}{4\cos\alpha_i} + \frac{\frac{|e_i|^2}{4} - |e_i|^2}{\frac{|e_i|\sin\alpha_i}{\cos\alpha_i}}}{\tan 89.5^\circ + \frac{|e_i|\cos\alpha_i - |e_i|}{|e_i|\sin\alpha_i}} = \frac{\tan 89.5^\circ + \frac{\tan\alpha_i}{4} - \frac{3}{4\tan\alpha_i}}{\tan 89.5^\circ - \frac{1 - \cos\alpha_i}{\sin\alpha_i}}|e_i|$. Next, exploiting $\sin 60^\circ \leq \sin\alpha_i \leq \sin 61^\circ$, $\tan 60^\circ \leq \tan\alpha_i \leq \tan 61^\circ$, and $\cos 61^\circ \leq \cos\alpha_i \leq \cos 60^\circ$ (which hold by assumption and by Lemma 2), we get $x \leq \frac{\tan 89.5^\circ + \frac{\tan 61^\circ}{4} - \frac{3}{4\tan 61^\circ}}{\tan 89.5^\circ - \frac{1 - \cos 61^\circ}{\sin 60^\circ}}|e_i| < 1.0056|e_i|$. Hence, the $y$-coordinate of $h^{|}(v, u_{i+1}, \overline{\alpha_i})$ is $y \leq \tan 89.5^\circ(1.0056|e_i| - |e_i|) < 0.642|e_i|$. Finally, the distance between $h^{|}(v, u_{i+1}, \overline{\alpha_i})$ and $u_{i+1}$ is at most $\sqrt{(0.642)^2 + (0.0056)^2}|e_i| < 0.6421|e_i|$, thus proving the claim in the case in which $|e| \geq |e_i|$.

Suppose that $|e| \leq |e_i|$. Then, by Lemma 5, $e \geq 2|e_i|\cos\alpha_i$. Using the last two inequalities we get $x \leq \frac{\tan 89.5^\circ|e_i| + \frac{|e_i|\sin\alpha_i}{2} + \frac{|e_i|^2\cos^2\alpha_i - |e_i|^2}{2|e_i|\sin\alpha_i}}{\tan 89.5^\circ + \frac{2|e_i|\cos^2\alpha_i - |e_i|}{2|e_i|\sin\alpha_i\cos\alpha_i}}$. Next, exploiting $\sin 60^\circ \leq \sin\alpha_i \leq \sin 61^\circ$, $\tan 60^\circ \leq \tan\alpha_i \leq \tan 61^\circ$, and $\cos 61^\circ \leq \cos\alpha_i \leq \cos 60^\circ$ (which hold by assumption and by Lemma 2), we get $x \leq \frac{\tan 89.5^{\circ} + \frac{\sin 61^{\circ}}{2} - \frac{1-\cos^2 60^{\circ}}{2\sin 61^{\circ}}}{\tan 89.5^{\circ} - \frac{1-2\cos^2 61^{\circ}}{2\sin 60^{\circ}\cos 61^{\circ}}}|e_i| < 1.0057|e_i|$. Hence, the $y$-coordinate of $h^{|}(v, u_{i+1}, \overline{\alpha_i})$ is $y \leq \tan 89.5^{\circ}(1.0057|e_i| - |e_i|) < 0.654|e_i|$. Finally, the distance between $h^{|}(v, u_{i+1}, \overline{\alpha_i})$ and $u_{i+1}$ is at most $\sqrt{(0.654)^2 + (0.0057)^2}|e_i| < 0.655|e_i|$, thus proving the claim in the case in which $|e| \leq |e_i|$. □

By Lemma 12 (where $u_{i+1}$, $k(p,m)$, and $l_{\tan 89.5^{\circ}}$ replace $u_i$, $k(v,m)$, and $l(\overline{\alpha_i})$ ) the distance between $u_{i+1}$ and the highest intersection point of $k(p,m)$ and $l_{\tan 89.5^{\circ}}$ is at least $1.604|e_{i-1}|$. Hence, $q$ is not above $k(p,m)$, as otherwise it is above $l^{|}_{pu_{i+1}}$, thus contradicting Lemma 1, and is not inside $k(p,m)$, again by Lemma 1. Then $q$ is below $k(p,m)$, and hence $|e^*|$ is at most $|u_{i+1}t|$. Then, we have:

**Claim 3** *If $|e| \geq |e_i|$, it holds $\frac{|\overline{u_{i+1}t}|}{|e_i|} < 0.056$; if $|e_i| \geq |e|$, it holds $\frac{|\overline{u_{i+1}t}|}{|e_i|} < 0.0723$.*

**Proof:** Suppose that $|e| \geq |e_i|$. Then we have $m = |e|$. The $x$-coordinate of $t$ satisfies $(\tan^2 89.5^{\circ} + 1)x^2 - 2(|e_i|\tan^2 89.5^{\circ} + |e|\tan 89.5^{\circ}\sin\alpha_i + |e|\cos\alpha_i)x + |e_i|^2\tan^2 89.5^{\circ} + 2|e_i||e|\tan 89.5^{\circ}\sin\alpha_i + |e|^2 - |e|^2 = 0$ (see the proof of Lemma 11), that yields $x = \frac{|e_i|\tan^2 89.5^{\circ} + |e|\tan 89.5^{\circ}\sin\alpha_i + |e|\cos\alpha_i}{\tan^2 89.5^{\circ}+1} \pm \frac{\sqrt{(|e_i|\tan^2 89.5^{\circ} + |e|\tan 89.5^{\circ}\sin\alpha_i + |e|\cos\alpha_i)^2 - (\tan^2 89.5^{\circ}+1)(|e_i|^2\tan^2 89.5^{\circ} + 2|e_i||e|\tan 89.5^{\circ}\sin\alpha_i)}}{\tan^2 89.5^{\circ}+1}$. Simplifying the last equation and observing that the $x$-coordinate of $t$ is the smallest of the two $x$-coordinates solving such an equation, we get $x = \frac{|e_i|\tan^2 89.5^{\circ} + |e|\tan 89.5^{\circ}\sin\alpha_i + |e|\cos\alpha_i}{\tan^2 89.5^{\circ}+1} - \frac{\sqrt{|e|^2\tan^2 89.5^{\circ}\sin^2\alpha_i + |e|^2\cos^2\alpha_i + 2|e_i||e|\tan^2 89.5^{\circ}\cos\alpha_i + 2|e|^2\tan 89.5^{\circ}\sin\alpha_i\cos\alpha_i - |e_i|^2\tan^2 89.5^{\circ} - 2|e_i||e|\tan 89.5^{\circ}\sin\alpha_i}}{\tan^2 89.5^{\circ}+1}$. Using $|e_i| \leq |e| \leq \frac{|e_i|}{2\cos\alpha_i}$, $\cos 61^{\circ} \leq \cos\alpha_i \leq \cos 60^{\circ}$, $\sin 60^{\circ} \leq \sin\alpha_i \leq \sin 61^{\circ}$, and $\tan 60^{\circ} \leq \tan\alpha_i \leq \tan 61^{\circ}$ we get $x \leq \frac{|e_i|\tan^2 89.5^{\circ} + \frac{|e_i|\tan 89.5^{\circ}\tan 61^{\circ}}{2} + \frac{|e_i|}{2}}{\tan^2 89.5^{\circ}+1} - \frac{|e_i|\sqrt{\tan^2 89.5^{\circ}\sin^2 60^{\circ} + \cos^2 61^{\circ} + 2\tan^2 89.5^{\circ}\cos 61^{\circ} + 2\tan 89.5^{\circ}\sin 60^{\circ}\cos 61^{\circ} - \tan^2 89.5^{\circ} - \tan 89.5^{\circ}\tan 61^{\circ}}}{\tan^2 89.5^{\circ}+1} < \frac{13234.420437 - 96.637136}{13131.5587}|e_i| < 1.00048|e_i|$. Hence, the $y$-coordinate of $t$ is at most $\tan 89.5^{\circ}(1.00048|e_i| - |e_i|) < 0.055|e_i|$. It follows that $\frac{|\overline{u_{i+1}t}|}{|e_i|} \leq \sqrt{0.00048^2 + 0.055^2} < 0.056$.

Suppose that $|e_i| \geq |e|$. Then we have $m = |e_i|$. The $x$-coordinate of $t$ satisfies $(\tan^2 89.5^{\circ} + 1)x^2 - 2(|e_i|\tan^2 89.5^{\circ} + |e|\tan 89.5^{\circ}\sin\alpha_i + |e|\cos\alpha_i)x + |e_i|^2\tan^2 89.5^{\circ} + 2|e_i||e|\tan 89.5^{\circ}\sin\alpha_i + |e|^2 - |e_i|^2 = 0$ (see the proof of Lemma 11). Solving with respect to $x$, observing that the $x$-coordinate of $t$ is the smallest of the two $x$-coordinates solving the previous equation, and using $|e| \leq |e_i| \leq \frac{|e|}{2\cos\alpha_i}$, $\cos 61^{\circ} \leq \cos\alpha_i \leq \cos 60^{\circ}$, $\sin 60^{\circ} \leq \sin\alpha_i \leq \sin 61^{\circ}$, and $\tan 60^{\circ} \leq \tan\alpha_i \leq \tan 61^{\circ}$, analogously to the case in which $|e| \geq |e_i|$ we get $x < 1.00063|e_i|$. Hence, the $y$-coordinate of $t$ is at most $\tan 89.5^{\circ}(1.00063|e_i| - |e_i|) < 0.0722|e_i|$. It follows that $\frac{|\overline{u_{i+1}t}|}{|e_i|} \leq \sqrt{0.00063^2 + 0.0722^2} < 0.0723$. □

Such a claim concludes the proof of the lemma. □

The next lemma asserts that if $\beta_i$ and $\beta'_i$ are large, then all the edges incident to $u_i$ have about the same length. Denote by $e_{i-1}$, $e_i^1$, $e_i^2$, $e_i^3$, and $e_i^4$ the clockwise order of the edges incident to $u_i$, where $\beta_i$ and $\beta'_i$ are both incident to $e_{i-1}$.

**Lemma 16** *Suppose that $\beta_i, \beta'_i \geq 89.5°$. Then $\max\{e_i^2, e_i^3, e_i^4\} \leq \frac{|e_i^1|}{2\cos(240^{\circ} - (\beta_i + \beta'_i))} \leq 1.032|e_i^1|$.*

**Proof:** Denote by $\gamma_1$, $\gamma_2$, and $\gamma_3$ the angles delimited by edges $e_i^1$ and $e_i^2$, by edges $e_i^2$ and $e_i^3$, and by edges $e_i^3$ and $e_i^4$, respectively. Observe that $\beta_i + \beta'_i \geq 179^\circ$, by the lemma's hypotheses, hence $\gamma_1 + \gamma_2 + \gamma_3 \leq 181^\circ$. By Lemma 2, $\gamma_1, \gamma_2, \gamma_3 \geq 60^\circ$, hence we have $\beta_i + \beta'_i \leq 180^\circ$, $\gamma_i \leq 240^\circ - (\beta_i + \beta'_i)$, with $i \in \{1, 2, 3\}$, and $\gamma_i + \gamma_j \leq 300^\circ - (\beta_i + \beta'_i)$, with $i, j \in \{1, 2, 3\}$ and $i \neq j$. Further, by Lemma 5, we have $|e_i^2| \leq \frac{|e_i^1|}{2\cos\gamma_1}$, $|e_i^3| \leq \frac{|e_i^1|}{4\cos\gamma_1\cos\gamma_2}$, and $|e_i^4| \leq \frac{|e_i^1|}{8\cos\gamma_1\cos\gamma_2\cos\gamma_3}$.

The second inequality directly comes from the fact that $\cos(240^\circ - (\beta_i + \beta'_i)) \geq \cos 61^\circ > 0.484$, hence $\frac{|e_i^1|}{2\cos(240^\circ - (\beta_i + \beta'_i))} \leq 1.032|e_i^1|$.

We prove the first inequality. First, $|e_i^2| \leq \frac{|e_i^1|}{2\cos(240^\circ - (\beta_i + \beta'_i))}$ directly comes from $|e_i^2| \leq \frac{|e_i^1|}{2\cos\gamma_1}$ and from $\gamma_1 \leq 240^\circ - (\beta_i + \beta'_i)$.

Second, to prove $|e_i^3| \leq \frac{|e_i^1|}{2\cos(240^\circ - (\beta_i + \beta'_i))}$, we use $|e_i^3| \leq \frac{|e_i^1|}{4\cos\gamma_1\cos\gamma_2}$ and we argue that $\frac{|e_i^1|}{4\cos\gamma_1\cos\gamma_2} \leq \frac{|e_i^1|}{2\cos(240^\circ - (\beta_i + \beta'_i))}$. Observe that $\frac{|e_i^1|}{4\cos\gamma_1\cos\gamma_2} \leq \frac{|e_i^1|}{2\cos(240^\circ - (\beta_i + \beta'_i))}$ is equivalent to $2\cos\gamma_1\cos\gamma_2 \geq \cos(240^\circ - (\beta_i + \beta'_i))$. Hence, we study the minimum value of $\cos\gamma_1\cos\gamma_2$. Observe that $\cos\gamma_i$ is a function decreasing with $\gamma_i$ when $0 \leq \gamma_i \leq 90^\circ$, hence, in order to minimize $\cos\gamma_1\cos\gamma_2$, we can assume that $\gamma_3 = 60^\circ$ and thus $\gamma_2 = (300 - \beta_i - \beta'_i) - \gamma_1$. The derivative of $\cos\gamma_1\cos((300 - \beta_i - \beta'_i) - \gamma_1)$ with respect to $\gamma_1$ is $-\sin\gamma_1\cos((300 - \beta_i - \beta'_i) - \gamma_1) + \cos\gamma_1\sin((300 - \beta_i - \beta'_i) - \gamma_1) = \sin((300 - \beta_i - \beta'_i) - 2\gamma_1)$. Hence, such a derivative is positive when $60^\circ \leq \gamma_1 < \frac{300 - \beta_i - \beta'_i}{2}$, is null when $\gamma_1 = \frac{300 - \beta_i - \beta'_i}{2}$, and is negative when $\frac{300 - \beta_i - \beta'_i}{2} < \gamma_1 \leq (240 - \beta_i - \beta'_i)$. Thus, the minimum of $\cos\gamma_1\cos\gamma_2$ is achieved either when $\gamma_1 = 60^\circ$ and $\gamma_2 = 240 - \beta_i - \beta'_i$ or when $\gamma_1 = 240 - \beta_i - \beta'_i$ and $\gamma_2 = 60^\circ$. Thus, we get $2\cos\gamma_1\cos\gamma_2 \geq \cos(240^\circ - (\beta_i + \beta'_i))$.

Third, to prove that $|e_i^4| \leq \frac{|e_i^1|}{2\cos(240^\circ - (\beta_i + \beta'_i))}$, we use $|e_i^4| \leq \frac{|e_i^1|}{8\cos\gamma_1\cos\gamma_2\cos\gamma_3}$ and we argue that $\frac{|e_i^1|}{8\cos\gamma_1\cos\gamma_2\cos\gamma_3} \leq \frac{|e_i^1|}{2\cos(240^\circ - (\beta_i + \beta'_i))}$. Similarly to the previous proof, it suffices to show that $4\cos\gamma_1\cos\gamma_2\cos\gamma_3 \geq \cos(240^\circ - (\beta_i + \beta'_i))$. Hence, we study the minimum value of $\cos\gamma_1\cos\gamma_2\cos\gamma_3$. Suppose that $\gamma_3$ is fixed to be any angle such that $60^\circ \leq \gamma_3 \leq 240^\circ - (\beta_i + \beta'_i)$. Then, analogously to the previous proof, it can be shown that the minimum value of $\cos\gamma_1\cos\gamma_2$ is achieved when one between $\gamma_1$ and $\gamma_2$, say $\gamma_1$, is $60^\circ$, while the other one, say $\gamma_2$, is $300 - \beta_i - \beta'_i - \gamma_3$. Hence, $\cos\gamma_1\cos\gamma_2\cos\gamma_3$ is minimized when $\cos\gamma_2\cos\gamma_3$ is minimized. Then, analogously to the previous proof, it can be shown that the minimum value of $\cos\gamma_2\cos\gamma_3$ is achieved when one between $\gamma_2$ and $\gamma_3$, say $\gamma_2$, is $60^\circ$, while the other one, say $\gamma_3$, is $240 - \beta_i - \beta'_i$. Thus, we get $4\cos\gamma_1\cos\gamma_2\cos\gamma_3 \geq \cos(240^\circ - (\beta_i + \beta'_i))$. □

The previous two lemmata, together with Lemma 6, imply the following.

**Corollary 1** *Suppose that $\alpha_{i-1} \leq 61^\circ$ and that $\beta'_i \geq 89.5^\circ$. Then, all the edges incident to $u_i$ and different from $e_{i-1}$ have length at most $0.1|e_{i-1}|$.*

# 4 The proof of the area bound

We prove that any MST embedding of $T^*$ is such that, for each backbone vertex $u_i$ of $C^*$, the outgoing angles of $u_i$ are either both small or one small and one large. We derive a $2^{\Omega(n)}$ lower bound on the area requirements of any MST embedding of $T^*$. Refer to the notation of Sect. 3. Let $k$ be the number of backbone vertices of $C^*$.

**Lemma 17** *For each $0 \leq i \leq k-2$, one of the following holds: (Condition 1): $\alpha_i, \alpha'_i \leq 61^\circ$; (Condition 2): $\alpha_i \geq 89.5^\circ$, $\alpha'_i \leq 61^\circ$, and $Cl(u_i)$ is in a bounded region $R_i$ that is a subset of a wedge $W_i$ with angle $1^\circ$ centered at $u_i$; (Condition 3): $\alpha'_i \geq 89.5^\circ$, $\alpha_i \leq 61^\circ$, and $Ccl(u_i)$ is in a bounded region $R_i$ that is a subset of a wedge $W_i$ with angle $1^\circ$ centered at $u_i$.*

**Proof:** The proof is by induction on $i$. In the base case $i = 0$ and, by Lemma 8, $\alpha_0, \alpha'_0 \leq 61^\circ$, thus Condition 1 holds. Next we discuss the inductive case.

Suppose that Condition 1 holds for $i$. By Lemma 6, we have $\beta_{i+1}, \beta'_{i+1} \geq 89.5^\circ$. By Corollary 1, all the edges incident to $u_{i+1}$ and different from $e_i$ have length at most $|e_i|/10$. By Lemma 2, each of the angles incident to $u_{i+1}$ and different from $\beta_{i+1}$ and $\beta'_{i+1}$ is at most $61^\circ$. Hence, if $e_{i+1}$ is in position 2 or 3, then Condition 1 holds for $i+1$. If $e_{i+1}$ is in position 1 (that is $\alpha_{i+1} = \beta_{i+1}$), then $\alpha'_{i+1} \leq 61^\circ$. Moreover, by Lemma 6, $\beta'_{i+2} \geq 89.5^\circ$. Then, all the conditions of Lemma 13 are satisfied, namely $\alpha_i \leq 61^\circ$, $\beta'_{i+1}, \beta'_{i+2} \geq 89.5^\circ$, and $|e_{i+1}| \leq |e_i|/10$. Hence, $Cl(u_{i+1})$ is in a bounded region $R_{i+1}$ that is a subset of $W_{i+1}$ and thus Condition 2 holds for $i+1$. If $e_{i+1}$ is in position 4, then an analogous proof reshows that Condition 3 holds for $i+1$.

Suppose that Condition 2 holds for $i$ (the case in which Condition 3 holds for $i$ can be discussed symmetrically). By Lemma 6, $\beta'_{i+1} \geq 89.5^\circ$. Hence, all the conditions of Lemma 14 are satisfied, namely $\alpha_i \geq 89.5^\circ$, $\beta'_{i+1} \geq 89.5^\circ$, and $Cl(u_i)$ is in a bounded region $R_i$ that is a subset of a wedge $W_i$ with angle $1^\circ$ centered at $u_i$. It follows that $\beta_{i+1} \geq 89.5^\circ$ and $Cl(u_{i+1})$ is in a bounded region $R_{i+1}$ that is a subset of a wedge $W_{i+1}$ with angle $1^\circ$ centered at $u_{i+1}$. By Lemma 2, each angle incident to $u_{i+1}$ and different from $\beta_{i+1}$ and $\beta'_{i+1}$ is at most $61^\circ$. Thus, if $e_{i+1}$ is in position 2 or 3, then Condition 1 holds for $i+1$, and if $e_{i+1}$ is in position 1, then Condition 2 holds for $i+1$. Suppose that $e_{i+1}$ is in position 4. Since each angle incident to $u_{i+1}$ and different from $\beta_{i+1}$ and $\beta'_{i+1}$ is at most $61^\circ$, it holds $\alpha_{i+1} \leq 61^\circ$ and then, by Lemma 6, $\beta_{i+2} \geq 89.5^\circ$. Since $\beta_{i+1}, \beta'_{i+1} \geq 89.5^\circ$, by Corollary 1 all the edges incident to $u_{i+1}$ and different from $e_i$ have length at most $|e_i|/10$. Then, all the conditions of the symmetric of Lemma 13 are satisfied, namely $\alpha'_i \leq 61^\circ$, $\beta_{i+1}, \beta_{i+2} \geq 89.5^\circ$, and $|e_{i+1}| \leq |e_i|/10$. Hence, $Ccl(u_{i+1})$ is in a bounded region $R_{i+1}$ that is a subset of $W_{i+1}$ and thus Condition 3 holds for $i+1$. □

**Theorem 1** *Any MST embedding of $T^*$ has $2^{\Omega(n)}$ area.*

**Proof:** Since the complete tree $T_c$ has constant degree and constant height, then each caterpillar, and in particular $C^*$, has $k = \Omega(n)$ backbone vertices. By Lemmata 6, 14, and 17, the incoming angles $\beta_i$ and $\beta'_i$ are both larger than $89.5^\circ$, for each $1 \leq i \leq k-1$. By Corollary 1, $|e_{i+1}| \leq \frac{|e_i|}{10}$, for each $0 \leq i \leq k-1$. Hence $\frac{|e_1|}{|e_k|} \geq 10^{k-1} = 2^{\Omega(n)}$. The theorem follows by observing that, in any MST embedding of the root of $T_c$ and of its children, both dimensions have size at least $\sin 30^\circ = 0.5$. □

# 5 Conclusions

In this paper we have shown trees requiring exponential area in any MST embedding, thus settling a 20-years-old problem proposed by Monma and Suri [12]. The actual conjecture of Monma and Suri states that both coordinate directions of any MST embedding of certain trees have exponential length. However, we believe that some further geometric considerations on the tree $T^*$ we presented in this paper can lead to completely settle the Monma and Suri's conjecture. Observe that the area requirements of the MST embeddings constructed by the algorithm presented by Monma and Suri is $2^{\Omega(n^2)}$, while no $2^{O(n)}$-area MST embeddings are known to exist for all $n$-vertex degree-5 trees. We believe that such a gap can be closed by further improving our exponential lower bound, as in the following.

**Conjecture 1** *Every MST embedding of $T^*$ has $2^{\Omega(n^2)}$ area.*

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