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Monotone Drawings of Graphs

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ABSTRACT

We study a new standard for visualizing graphs: A monotone drawing is a straight-line drawing such that, for every pair of vertices, there exists a path that monotonically increases with respect to some direction. We show algorithms for constructing monotone planar drawings of trees and biconnected planar graphs, we study the interplay between monotonicity, planarity, and convexity, and we outline a number of open problems and future research directions.

1 Introduction

A traveler that consults a road map to find a route from a site u to a site v would like to easily spot at least one path connecting u and v. Such a task is harder if each path from u to v on the map has legs moving away from v. Travelers rotate maps to better perceive their content. Hence, even if in the original orientation of the map all the paths from u to v have annoying back and forth legs, the traveler might be happy to find at least one orientation where a path from u to v smoothly flows from left to right.

Leaving the road map metaphora for the Graph Drawing terminology, we say that a path P in a straight-line drawing of a graph is *monotone* if there exists a line l such that the orthogonal projections of the vertices of P on l appear along l in the order induced by P. A straight-line drawing of a graph is *monotone* if it contains at least one monotone path for each pair of vertices. Having at disposal a monotone drawing (map), for each pair of vertices u and v a user (traveler) can find a rotation of the drawing such that there exists a path from u to v always increasing in the x-coordinate.

In a monotone drawing each monotone path is monotone with respect to a different line. *Upward drawings* [6, 9] are related to monotone drawings, as in an upward drawing every directed path is monotone. Even more related to monotone drawings are *greedy drawings* [12, 11, 1]. Namely, in a greedy drawing, between any two vertices a path exists such that the Euclidean distance from an intermediate vertex to the destination decreases at every step, while, in a monotone drawing, between any two vertices a path and a line l exist such that the Euclidean distance from the projection of an intermediate vertex on l to the projection of the destination on l decreases at every step.

Monotone drawings have a strict correlation with an important problem in Computational Geometry: Arkin, Connelly, and Mitchell [2] studied how to find monotone trajectories connecting two given points in the plane avoiding convex obstacles. As a corollary, they proved that every planar convex drawing is monotone. Hence, the graphs admitting a convex drawing [5] have a planar monotone drawing. Such a class of graphs is a super-class (sub-class) of the triconnected (biconnected) planar graphs.

In this paper we first deal with trees (Sect. 4). We prove several properties relating the monotonicity of a tree drawing to its planarity and "convexity" [4]. Moreover, we show two algorithms for constructing monotone planar grid drawings of trees. The first one constructs drawings lying on a grid of size $O(n^{1.6}) \times O(n^{1.6})$. The second one has a better area requirement, namely $O(n^3)$, but a worse $\Omega(n)$ aspect ratio.

The existence of monotone drawings of trees allows to construct a monotone drawing of any graph G by drawing any of its spanning trees. However, the obtained monotone drawing could be non-planar even if G is a planar graph. Motivated by this and since every triconnected planar graph admits a planar monotone drawing, we devise an algorithm to construct planar monotone drawings of biconnected planar graphs (Sect. 5). Such an algorithm exploits the SPQR-tree decomposition of a biconnected planar graph.

We conclude the paper with several open problems (Sect. 6).

2 Definitions and Preliminaries

A *straight-line drawing* of a graph is a mapping of each vertex to a distinct point of the plane and of each edge to a segment connecting its endpoints. A drawing is *planar* if the segments representing its edges do not cross but, possibly, at common endpoints. A graph is *planar* if it admits a planar drawing. A planar drawing partitions the plane into topologically connected regions, called *faces*. The unbounded face is the *outer face*. A *strictly convex drawing* (resp. a *(non-strictly) convex drawing*) is a straight-line planar drawing in which each face is delimited by a strictly (resp. non-strictly) convex polygon.

We denote by $P(v_1, v_m)$ a path between vertices v_1 and v_m. A graph G is *connected* if every pair of vertices is connected by a path and is *biconnected* (resp. *triconnected*) if removing any vertex (resp. any two vertices) leaves G connected. A *subdivision* of G is the graph obtained by replacing each edge of G with a path. A *subdivision of a drawing* Γ of G is a drawing Γ' of a subdivision G' of G such that, for every edge (u, v) of G that has been replaced by a path $P(u, v)$ in G', u and v are drawn at the same point in Γ and in Γ', and all the vertices of $P(u, v)$ lie on the segment between u and v.

Let p be a point in the plane and l an half-line starting at p. The *slope* of l, denoted by $slope(l)$, is the angle spanned by a counter-clockwise rotation that brings a horizontal half-line starting at p and directed towards increasing x-coordinates to coincide with l. We consider slopes that are equivalent modulo 2π as the same slope (e.g., $\frac{3}{2}\pi$ is regarded as the same slope as -$\frac{\pi}{2}$). Let Γ be a drawing of a graph G and (u, v) an edge of G. The half-line starting at u and passing through v, denoted by $d(u, v)$, is the *direction* of (u, v). The *slope* of (u, v), denoted by $slope(u, v)$, is the slope of $d(u, v)$. Observe that $slope(u, v) = slope(v, u) - \pi$. When comparing directions and their slopes, we assume that they are applied at the origin of the axes. An edge (u, v) is *monotone* with respect to a half-line l if it has a "positive projection" on l, i.e., if $slope(l) - \frac{\pi}{2} < slope(u, v) < slope(l) + \frac{\pi}{2}$. A path $P(u_1, u_n) = (u_1, \dots, u_n)$ is *monotone with respect to a half-line* l if (u_i, u_{i+1}) is monotone with respect to l, for $i = 1, \dots, n-1$; $P(u_1, u_n)$ is *monotone* if there exists a half-line l such that $P(u_1, u_n)$ is monotone with respect to l. Observe that if a path $P(u_1, u_n) = (u_1, \dots, u_n)$ is monotone with respect to l, then the orthogonal projections on l of $u_1, \dots, u_n$ appear in this order along l. A drawing Γ of a graph G is *monotone* if, for each pair of vertices u and v in G, there exists a monotone path $P(u, v)$ in Γ. Observe that monotonicity implies connectivity.

The *Stern-Brocot tree* [13, 3] is an infinite tree whose nodes are in bijective mapping with the irreducible positive rational numbers. Refer to Fig. 1. The Stern-Brocot tree $\mathcal{SB}$ has two nodes 0/1 and 1/0 that are connected to the same node 1/1, where 1/1 is the right child of 0/1 and 1/1 is the left child of 1/0. An ordered binary tree is then rooted at 1/1 as follows. Consider a node y/x of the tree. The left child of y/x is the node $(y+y')/(x+x')$, where y'/x' is the ancestor of y/x that is closer to y/x (in terms of graph-theoretic distance in $\mathcal{SB}$) and that has y/x in its right subtree. The right child of y/x is the node $(y+y'')/(x+x'')$, where y''/x'' is the ancestor of y/x that is closer to y/x and that has y/x in its left subtree. The *first level* of $\mathcal{SB}$ is composed of node 1/1. The *i-th level* of $\mathcal{SB}$ is composed of the children of the nodes of the $(i-1)$-th level of $\mathcal{SB}$. The following property of the Stern-Brocot tree is well-known and easy to observe:

Property 1 *The sum of the numerators of the elements of the i-th level of $\mathcal{SB}$ is 3^{i-1} and the sum of the denominators of the elements of the i-th level of $\mathcal{SB}$ is 3^{i-1}.*

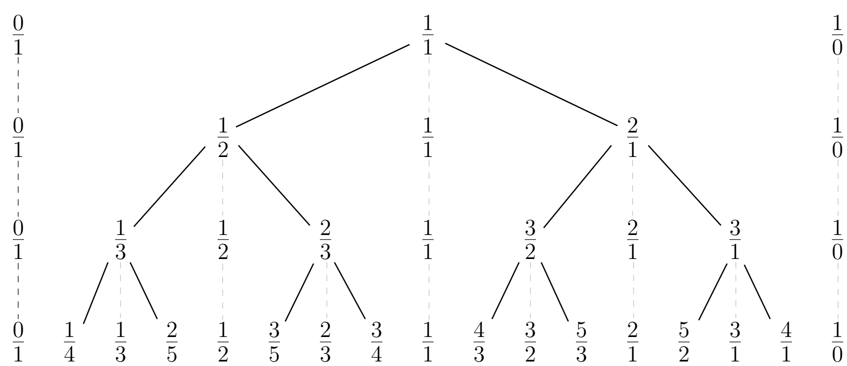

Figure 1: Construction of the Stern-Brocot tree.

2.1 The SPQR-Tree Decomposition

To decompose a biconnected graph into its triconnected components, we use the *SPQR-tree*, a data structure introduced by Di Battista and Tamassia [7, 8].

A graph is *st-biconnectible* if adding edge (s,t) to it yields a biconnected graph. Let G be an st-biconnectible graph. A *separation pair* of G is a pair of vertices whose removal disconnects the graph. A *split pair* of G is either a separation pair or a pair of adjacent vertices. A *maximal split component* of G with respect to a split pair $\{u,v\}$ (or, simply, a maximal split component of $\{u,v\}$) is either an edge (u,v) or a maximal subgraph G' of G such that G' contains u and v, and $\{u,v\}$ is not a split pair of G'. A vertex $w \neq u,v$ belongs to exactly one maximal split component of $\{u,v\}$. We call *split component* of $\{u,v\}$ the union of any number of maximal split components of $\{u,v\}$.

In the paper, we assume that any SPQR-tree of a graph G is rooted at one edge of G, called *reference edge*.

The rooted SPQR-tree $\mathcal{T}$ of a biconnected graph G, with respect to a reference edge e, describes a recursive decomposition of G induced by its split pairs. The nodes of $\mathcal{T}$ are of four types: S, P, Q, and R. Their connections are called *arcs*, in order to distinguish them from the edges of G.

Each node μ of $\mathcal{T}$ has an associated st-biconnectible multigraph, called the *skeleton* of μ and denoted by $skel(\mu)$. Skeleton $skel(\mu)$ shows how the children of μ, represented by "virtual edges", are arranged into μ. The virtual edge in $skel(\mu)$ associated with a child node ν, is called the *virtual edge of ν in $skel(\mu)$*.

For each virtual edge e_i of $skel(\mu)$, recursively replace e_i with the skeleton $skel(\mu_i)$ of its corresponding child μ_i. The subgraph of G that is obtained in this way is the *pertinent graph* of μ and is denoted by $pert(\mu)$.

Given a biconnected graph G and a reference edge $e = (u', v')$, tree $\mathcal{T}$ is recursively defined as follows. At each step, a split component G^*, a pair of vertices $\{u, v\}$, and a node ν in $\mathcal{T}$ are given. A node μ corresponding to G^* is introduced in $\mathcal{T}$ and attached to its parent ν. Vertices u and v are the *poles* of μ and denoted by $u(\mu)$ and $v(\mu)$, respectively. The decomposition possibly recurs on some split components of G^*. At the beginning of the decomposition $G^* = G - \{e\}$, $\{u, v\} = \{u', v'\}$, and ν is a Q-node corresponding to e.

Base Case: If G^* consists of exactly one edge between u and v, then μ is a Q-node whose skeleton is G^* itself.

Parallel Case: If G^* is composed of at least two maximal split components $G_1, \dots, G_k$ $(k \geq 2)$ of G with respect to $\{u, v\}$, then μ is a P-node. Graph $skel(\mu)$ consists of k parallel virtual edges between u and v, denoted by $e_1, \dots, e_k$ and corresponding to $G_1, \dots, G_k$, respectively. The decomposition recurs on $G_1, \dots, G_k$, with $\{u, v\}$ as pair of vertices for every graph, and with μ as parent node.

Series Case: If G^* is composed of exactly one maximal split component of G with respect to $\{u, v\}$ and if G^* has cutvertices $c_1, \dots, c_{k-1}$ $(k \geq 2)$, appearing in this order on a path from u to v, then μ is an S-node. Graph $skel(\mu)$ is the path $e_1, \dots, e_k$, where virtual edge e_i connects c_{i-1} with c_i $(i = 2, \dots, k-1)$, e_1 connects u with c_1, and e_k connects c_{k-1} with v. The decomposition recurs on the split components corresponding to each of $e_1, e_2, \dots, e_{k-1}, e_k$ with μ as parent node, and with $\{u, c_1\}, \{c_1, c_2\}, \dots, \{c_{k-2}, c_{k-1}\}, \{c_{k-1}, v\}$ as pair of vertices, respectively.

Rigid Case: If none of the above cases applies, the purpose of the decomposition step is that of partitioning G^* into the minimum number of split components and recurring on each of them. We need some further definition. Given a maximal split component G' of a split pair $\{s, t\}$ of G^*, a vertex $w \in G'$ *properly belongs* to G' if $w \neq s, t$. Given a split pair $\{s, t\}$ of G^*, a maximal split component G' of $\{s, t\}$ is *internal* if neither u nor v (the poles of G^*) properly belongs to G', *external* otherwise. A *maximal split pair* $\{s, t\}$ of G^* is a split pair of G^* that is not contained into an internal maximal split component of any other split pair $\{s', t'\}$ of G^*. Let $\{u_1, v_1\}, \dots, \{u_k, v_k\}$ be the maximal split pairs of G^* $(k \geq 1)$ and, for $i = 1, \dots, k$, let G_i be the union of all the internal maximal split components of $\{u_i, v_i\}$. Observe that each vertex of G^* either properly belongs to exactly one G_i or belongs to some maximal split pair $\{u_i, v_i\}$. Node μ is an R-node. Graph $skel(\mu)$ is the graph obtained from G^* by replacing each subgraph G_i with the virtual edge e_i between u_i and v_i. The decomposition recurs on each G_i with μ as parent node and with $\{u_i, v_i\}$ as pair of vertices.

For each node μ of $\mathcal{T}$, the construction of $skel(\mu)$ is completed by adding a virtual edge (u, v) representing the rest of the graph.

The SPQR-tree $\mathcal{T}$ of a graph G with n vertices and m edges has m Q-nodes and $O(n)$ S-, P-, and R-nodes. Also, the total number of vertices of the skeletons stored at the nodes of $\mathcal{T}$ is $O(n)$. Finally, SPQR-trees can be constructed and handled efficiently. Namely, given a biconnected planar graph G, the SPQR-tree $\mathcal{T}$ of G can be computed in linear time [7, 8, 10].

3 Properties of Monotone Drawings

In this section we give some basic properties of monotone drawings.

Property 2 *Any sub-path of a monotone path is monotone.*

Property 3 *A path $P(u_1, u_n) = (u_1, u_2, \ldots, u_n)$ is monotone if and only if it contains two edges e_1 and e_2 such that the closed wedge centered at the origin of the axes, delimited by the two half-lines $d(e_1)$ and $d(e_2)$, and having an angle smaller than π, contains all the half-lines $d(u_i, u_{i+1})$, for $i = 1, \ldots, n-1$.*

Edges e_1 and e_2 as in Property 3 are the *extremal edges* of $P(u_1, u_n)$. The closed wedge delimited by $d(e_1)$ and $d(e_2)$ and containing all the half-lines $d(u_i, u_{i+1})$, for $i = 1, \ldots, n-1$, is the *range* of $P(u_1, u_n)$ and is denoted by $range(P(u_1, u_n))$, while the closed wedge delimited by $d(e_1) - \pi$ and $d(e_2) - \pi$, and not containing $d(e_1)$ and $d(e_2)$, is the *opposite range* of $P(u_1, u_n)$ and is denoted by $opp(P(u_1, u_n))$.

Property 4 *The range of a monotone path $P(u_1, u_n)$ contains the half-line from u_1 through u_n.*

Proof: Draw the closed wedge $range(P(u_1, u_n))$ centered at u_1. Observe that a sequence of edges whose directions are $d(u_i, u_{i+1})$, with $i = 1, \ldots, n-1$, allow to reach from u_1 only points of $range(P(u_1, u_n))$. Hence, u_n and the half line from u_1 through u_n are contained into $range(P(u_1, u_n))$. □

Lemma 1 *Let $P(u_1, u_n) = (u_1, u_2, \ldots, u_n)$ be a monotone path and let (u_n, u_{n+1}) be an edge. Then, path $P(u_1, u_{n+1}) = (u_1, u_2, \ldots, u_n, u_{n+1})$ is monotone if and only if $d(u_n, u_{n+1})$ is not contained in $opp(P(u_1, u_n))$. Further, if $P(u_1, u_{n+1})$ is monotone, $range(P(u_1, u_n)) \subseteq range(P(u_1, u_{n+1}))$.*

Proof: Denote by e_1 and e_2 the extremal edges of $P(u_1, u_n)$.

If $d(u_n, u_{n+1})$ is in $opp(P(u_1, u_n))$, then no wedge having an angle smaller than π contains all of $d(e_1)$, $d(e_2)$, and $d(u_n, u_{n+1})$. By Property 3, $P(u_1, u_{n+1})$ is not monotone.

If $d(u_n, u_{n+1})$ is not in $opp(P(u_1, u_n))$, then it is contained either in $range(P(u_1, u_n))$, or in the smallest wedge delimited by $d(e_1)$ and by $d(e_2) - \pi$, or in the smallest wedge delimited by $d(e_2)$ and by $d(e_1) - \pi$. In the first case, $P(u_1, u_{n+1})$ is monotone by Property 3; further, we have that $range(P(u_1, u_{n+1})) = range(P(u_1, u_n))$. In the second case (the third one being symmetric), $d(u_n, u_{n+1})$ and $d(e_2)$ delimit a wedge having an angle smaller than π and all the half-lines $d(u_i, u_{i+1})$, for $i = 1, \ldots, n$, are contained in such a wedge (hence such a wedge contains $range(P(u_1, u_n))$; by Property 3, $P(u_1, u_{n+1})$ is monotone. □

Corollary 1 *Let $P(u_1, u_n) = (u_1, \ldots, u_n)$ and $P(u_n, u_{n+k}) = (u_n, \ldots, u_{n+k})$ be monotone paths. Then, path $P(u_1, u_{n+k}) = (u_1, \ldots, u_n, u_{n+1}, \ldots, u_{n+k})$ is monotone if and only if $range(P(u_1, u_n)) \cap opp(P(u_n, u_{n+k})) = \emptyset$. Further, if $P(u_1, u_{n+k})$ is monotone, $range(P(u_1, u_n)) \cup range(P(u_n, u_{n+k})) \subseteq range(P(u_1, u_{n+k}))$.*

The following properties relate monotonicity to planarity and convexity.

Property 5 *A monotone path is planar.*

Proof: Suppose, for a contradiction, that two edges (u_1, u_2) and (v_1, v_2) of a monotone path P intersect. Assume, without loss of generality, that the sub-path $P'(u_2, v_1)$ of P does not contain u_1 and v_2. By Property 2, $P'(u_2, v_1)$ is monotone. Also, by Property 4, $range(P'(u_2, v_1))$ contains $d(u_2, v_1)$.

Consider angles $\widehat{u_1 u_2 v_1}$ and $\widehat{u_2 v_1 v_2}$. As (u_1, u_2) and (v_1, v_2) intersect, we have $\widehat{u_1, u_2, v_1} + \widehat{u_2, v_1, v_2} < \pi$. Translate $d(u_1, u_2)$, $d(v_1, v_2)$, and $d(u_2, v_1)$ so that they are applied at the origin of the axes. Since $\widehat{u_1 u_2 v_1} + \widehat{u_2 v_1 v_2} < \pi$, we have that the wedge delimited by $d(u_1, u_2)$ and $d(v_1, v_2)$ and not containing $d(u_2, v_1)$ has an angle smaller than π. Further, the wedge delimited by $d(u_1, u_2)$ and $d(u_2, v_1)$ and not containing $d(v_1, v_2)$, and the wedge delimited by $d(v_1, v_2)$ and $d(u_2, v_1)$ and not containing $d(u_1, u_2)$ are easily shown to have angles smaller than π. Hence, it is not possible to find a wedge centered at the origin, having an angle smaller than π, and containing all of $d(u_1, u_2)$, $d(v_1, v_2)$, and $d(u_2, v_1)$. Thus, P is not monotone, a contradiction. □

Lemma 2 *[2] Any strictly convex drawing of a planar graph is monotone.*

When proving that every biconnected planar graph admits a planar monotone drawing, we will need to construct non-strictly convex drawings of graphs. Observe that any graph containing a degree-2 vertex does not admit a strictly convex drawing, while it might admit a non-strictly convex drawing. While not every non-strictly convex drawing is monotone, we can relate non-strict convexity and monotonicity:

Lemma 3 *Any non-strictly convex drawing of a graph such that each set of parallel edges forms a collinear path is monotone.*

Proof: First, observe that, starting from a convex drawing, several upward drawings can be defined. Namely, let Γ be a (not necessarily strictly) convex drawing of a graph G and let d be a direction which is not orthogonal to any edge of Γ. We associate with Γ the upward drawing Γ_d of the directed graph G_d where each edge of G has been directed so that the angle formed in Γ with d falls into the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Observe that the convexity of Γ ensures that G_d has a single source s_d and a single sink t_d (see Fig. 2(a)).

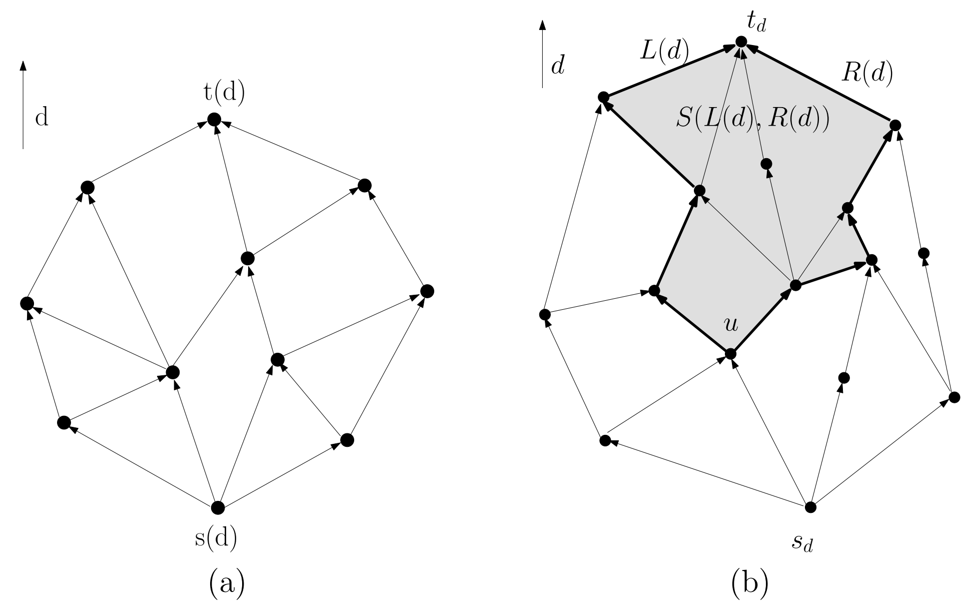

Figure 2: (a) The directed graph G_d induced by direction d on a convex drawing of a graph. (b) Paths $L(d)$ and $R(d)$ and the region $\mathcal{S}(L(d), R(d))$.

In order to prove the lemma, we directly apply to a non-strictly-convex drawing of a graph a technique which is similar to that used in [2] to prove the existence of a monotone path between two points of the plane in the presence of convex obstacles.

In order to find a monotone path from a vertex u to a vertex v in Γ, consider an arbitrary direction d and the upward drawing Γ_d of G_d. We are searching for a direction d^* such that there is a directed path from u to v in G_{d^*}.

Call *leftmost path* (resp. *rightmost path*) and denote by $L(d)$ (resp. $R(d)$) the directed path of Γ_d that starts from u, traverses Γ_d by taking the last (resp. first) exiting edge of each vertex in the counter-clockwise direction, and terminates on t_d (see Fig. 2(b) for an example). Since Γ is convex such paths always exist.

Call *slice of* $L(d)$ *and* $R(d)$ and denote by $\mathcal{S}(L(d), R(d))$, the closed region enclosed by the clockwise cycle composed of $L(d)$ and $R(d)$ traversed backward. Observe that, if $\mathcal{S}(L(d), R(d))$ is not collapsed into a single line (when $L(d) = R(d)$) $\mathcal{S}(L(d), R(d))$ is a bounded region. Intuitively, this is due to the fact that $L(d)$ always "precedes" $R(d)$ in the counter-clockwise direction (see Fig. 2(b)).

The convexity of Γ ensures that any vertex in $\mathcal{S}(L(d), R(d))$ is reachable by a path monotone with respect to d. Hence, if v is drawn into $\mathcal{S}(L(d), R(d))$ we are done. Otherwise, suppose that v is not drawn into $\mathcal{S}(L(d), R(d))$. We continuously rotate counter-clockwise d until we find a direction d' that produces a $\Gamma_{d'} \neq \Gamma_d$. Again, if v falls into $\mathcal{S}(L(d'), R(d'))$ we are done, otherwise, we carry on the rotation process until we find a direction d^* such that $\mathcal{S}(L(d^*), R(d^*))$ contains v. To prove that such a direction d^* always exists, it suffices to show that $L(d')$ belongs to $\mathcal{S}(L(d''), R(d''))$ for any two consecutive directions d' and d''. Intuitively, this corresponds to say that the union of all the slices (for all possible directions) covers G.

First observe that, as parallel edges form collinear paths in Γ, $G_{d''}$ differs from $G_{d'}$ in the fact that the direction of a single edge or a single collinear path is reversed.

Suppose that only one among $L(d')$ and $R(d')$ changes, that is, that either $L(d'') = L(d')$ or $R(d'') = R(d')$. In both cases $L(d')$ trivially belongs to $\mathcal{S}(L(d''), R(d''))$.

Suppose that both $L(d')$ and $R(d')$ change into $L(d'')$ and $R(d'')$. Consider the slice $\mathcal{S}(R(d''), L(d'))$ which is the intersection of slices $\mathcal{S}(L(d''), L(d'))$ and $\mathcal{S}(R(d''), R(d'))$. If $\mathcal{S}(R(d''), L(d'))$ is contained into $\mathcal{S}(R(d''), L(d''))$, then also $L(d')$ is contained into $\mathcal{S}(R(d''), L(d''))$ and the proof terminates. Otherwise, we show that all vertices of the slice $\mathcal{S}(R(d''), L(d'))$ are on its boundary. The proof is by induction. Let w be a node common to $R(d'')$ and $L(d')$ such that all vertices of $R(d'')$ and $L(d')$ from u to w are on the boundary of $\mathcal{S}(R(d''), L(d'))$ (the base case being when $w = u$). Denote w_r the next vertex of $R(d'')$ and w_l the next vertex of $L(d')$. If $w_r = w_l$ then it is trivial that all vertices of $R(d'')$ and $L(d')$ from u to $w_r = w_l$ are on the boundary of $\mathcal{S}(R(d''), L(d'))$. If $w_r \neq w_l$, then it can not be the case that w_r precedes w_l in the counter-clockwise order of the adjacency list of w. In fact, in this case $\mathcal{S}(R(d''), L(d'))$ is contained into $\mathcal{S}(R(d''), L(d''))$, a case that is already handled above. Hence, w_l precedes w_r in the counter-clockwise order of the adjacency list of w.

Since $R(d'')$ uses (w, w_r) instead of (w, w_l), the edge (w, w_l) must be directed from w to w_l in $G_{d'}$ and from w_l to w in $G_{d''}$. Also, (w, w_r) directly precedes (w, w_l) in the counter-clockwise order of the adjacency list of w. This is because only one edge (or sequence of collinear edges) changes its orientation from $G_{d'}$ to $G_{d''}$, and an edge exiting w between (w, w_r) and (w, w_l) would have been used by $R(d'')$. Hence, (w, w_r) and (w, w_l) are on the same face f. Also, since f can not have two sequences of edges that are parallel, $L(d')$ traverses the right border of face f until the sink $s_{d'}(f)$ of f in $G_{d'}$ is reached. Instead, $R(d'')$ traverses the left border of f until the sink $s_{d''}(f)$ of f in $G_{d''}$ is reached. Since only one sequence of parallel edges changes from $G_{d'}$ and $G_{d''}$ we have that $s_{d'}(f) = s_{d''}(f)$. This proves that all vertices of $R(d'')$ and $L(d')$ from u to $s(f)$ are on the boundary of $\mathcal{S}(R(d''), L(d'))$. □

Still on the relationship between convexity and monotonicity, we have:

Lemma 4 *Consider a strictly convex drawing Γ of a graph G. Let u, v, and w be three consecutive vertices incident to the outer face. Let d be any half-line that splits* *the angle $\widehat{uvw}$ into two angles smaller than $\frac{\pi}{2}$. Then, for each vertex t of G, there exists a path from v to t in Γ that is monotone with respect to d.*

Proof: For each vertex $x \in G$, consider the line l_x orthogonal to d and passing through x. Orient every edge (x, y) incident to x so that it enters x if y is in the half-plane delimited by l_x containing v and it exits x otherwise. Denote by G' the resulting graph. Observe that, if a directed path from a vertex x to a vertex y exists in G', then such a path is monotone with respect to d. Since d splits $\widehat{uvw}$ into two angles smaller than $\frac{\pi}{2}$, all the edges incident to v exit v in G'. Further, since Γ is convex, every vertex different from v has at least one entering edge. It follows that v is the unique source of G', which implies that, for each vertex t of G, there exists a path from v to t in G', and hence a path from v to t in Γ that is monotone with respect to d. □

Next, we provide a powerful tool for "transforming" monotone drawings.

Lemma 5 *An affine transformation applied to a monotone drawing yields a monotone drawing.*

Proof: Consider a path $P(u_1, u_n) = (u_1, u_2, \ldots, u_n)$ that is monotone with respect to an oriented line l. Suppose w.l.o.g. that l does not intersect $P(u_1, u_n)$. Construct a graph G whose vertices are $u_1, u_2, \ldots, u_n$ and their projections $v_1, v_2, \ldots, v_n$ on l, and whose edges are (u_i, u_{i+1}), (v_i, v_{i+1}), for $i = 1, \ldots, n-1$, and (u_i, v_i), for $i = 1, \ldots, n$. Observe that G is planar and that edges (u_i, v_i), for $i = 1, \ldots, n$, are parallel. The affine transformation preserves the planarity of G, the parallelism of edges (u_i, v_i), for $i = 1, \ldots, n$, and the collinearity of vertices v_i, for $i = 1, \ldots, n$. In particular, vertices $v_1, \ldots, v_n$ appear in this order on the line passing through them. Therefore, they appear in the same order when projected on any line perpendicular to (u_i, v_i), for $i = 1, \ldots, n$. The statement follows. □

4 Monotone Drawings of Trees

In this section we study monotone drawings of trees.

We present two interesting properties concerning monotone drawings of trees. The first one is about the relationship between monotonicity and planarity. Such a property directly descends from the fact that every monotone path is planar (by Property 5) and that in a tree there exists exactly one path between every pair of vertices.

Property 6 *Every monotone drawing of a tree is planar.*

The second property relates monotonicity and convexity. A *convex drawing* of a tree T [4] is a straight-line planar drawing such that replacing each edge between an internal vertex u and a leaf v with a half-line starting at u through v yields a partition of the plane into convex unbounded polygons. A convex drawing of a tree might not be monotone, because of the presence of two parallel edges. However, if we assume that such two edges do not exist, then a convex drawing is also monotone. Define a *strictly convex drawing* of a tree T as a straight-line planar drawing such that each set of parallel edges forms a collinear path and such that replacing every edge of T between an internal vertex u and a leaf v with a half-line starting at u through v yields a partition of the plane into convex unbounded polygons. We have the following:

Property 7 *Every strictly convex drawing of a tree is monotone.*

Proof: Consider any tree T. We can assume that T has no degree-2 vertex. Namely, in any strictly convex drawing Γ' of a tree T' with a degree-2 vertex u, the edges (v_1, u) and (v_2, u) incident to u lie both on the line through v_1 and v_2. Hence, u can be removed and edge (v_1, v_2) can be inserted in Γ' obtaining a drawing Γ'' of a tree T'' with one degree-2 vertex less. Further, Γ'' is strictly convex if and only if Γ' is.

Consider any strictly convex drawing Γ of a tree T with no degree-2 vertex. The drawing is monotone if and only if every path between two leaves is monotone. The necessity is trivial. For the sufficiency, observe that any path between two vertices that are not both leaves is a subpath of a path between two leaves. As the latter path is monotone, then the former path is monotone as well by Property 2.

Consider any path $P(u, v) = (u, z_1, z_2, \ldots, z_k, v)$ between two leaves u and v. Consider the edges (z_1, z'_1) and (z_1, z''_1) incident to z_1 and following (z_1, z_2) in clockwise and in counter-clockwise direction, respectively (see Fig. 3). Consider the half-lines l'_1 and l''_1 starting at z_1 through z'_1 and through z''_1, respectively. Define l'_1 and l''_1 as the *bounding half-lines for* z_2. Namely, all the edges incident to z_2 have a slope which is in the range defined by a counter-clockwise movement bringing (z_1, z'_1) to coincide with (z_1, z''_1), as otherwise one of the two polygons incident to edge (z_1, z_2) would not be convex. Hence, also edge (z_2, z_3) and the *bounding half-lines for* z_3, that are defined analogously as the ones for z_2, have slopes in such a range. Such an argument propagates throughout all the vertices of $P(u, v)$.

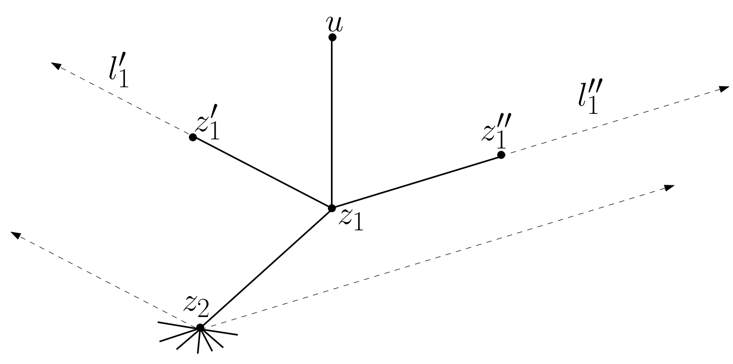

Figure 3: Edges (z_1, z'_1) and (z_1, z''_1) and half-lines l'_1 and l''_1.

Denote by $range(z_i)$ the range defined by a counter-clockwise movement bringing (z_i, z'_i) to coincide with (z_i, z''_i) and by $range(P(u, v), z_i)$ the smallest range containing all the slopes defined by the edges of $P(u,v)$ up to vertex z_i. We inductively prove the following statement that easily implies a proof of Property 7: (i) If $range(z_i)$ has an angle larger than or equal to π (see Fig. 4(a)), then $range(P(u,v),z_i)$ has an angle smaller than π and is a sub-range of $range(z_i)$; further, both the smallest range containing $range(P(u,v),z_i)$ and (z_i,z_i') and the smallest range containing $range(P(u,v),z_i)$ and (z_i,z_i'') have angles smaller than π; (ii) if $range(z_i)$ has an angle smaller than π (see Fig. 4(b)), then the range obtained as the union of $range(z_i)$ and $range(P(u,v),z_i)$ has an angle smaller than π.

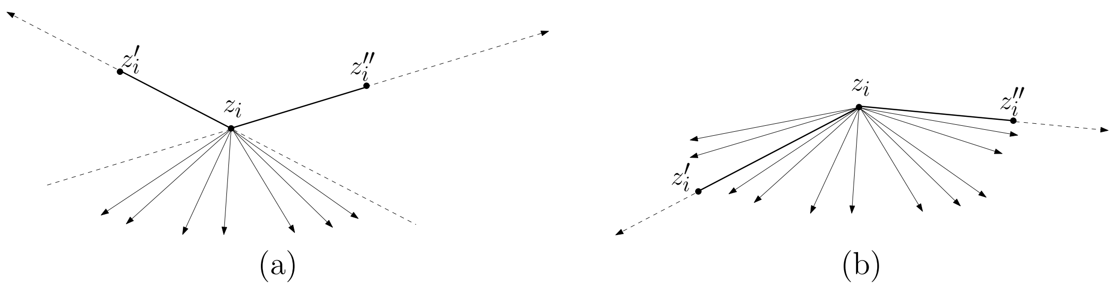

Figure 4: (a) $range(z_i)$ has an angle larger than or equal to π. (b) $range(z_i)$ has an angle smaller than π.

The statement is easily shown to be true for z_1. For the inductive case, some cases have to be distinguished.

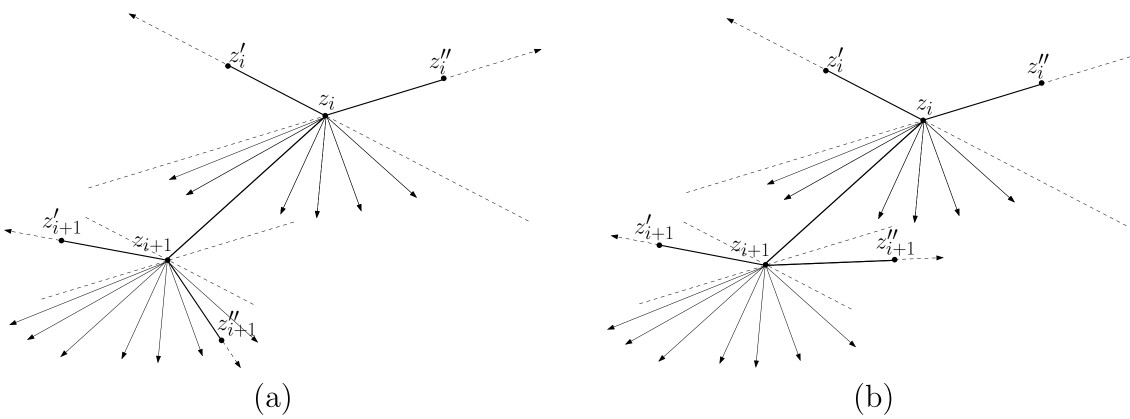

Figure 5: $range(z_i)$ has an angle larger than or equal to π, edge (z_i,z_{i+1}) is inside $range(P(u,v),z_i)$, and (a) $range(z_{i+1})$ has an angle smaller than π or (b) $range(z_{i+1})$ has an angle larger than or equal to π.

- Suppose that $range(z_i)$ has an angle larger than or equal to π.
  - If (z_i,z_{i+1}) is inside $range(P(u,v),z_i)$, then we have $range(P(u,v),z_{i+1}) = range(P(u,v),z_i)$.

- $*$ Suppose that $range(z_{i+1})$ has an angle smaller than π (see Fig. 5(a)). If $range(z_{i+1})$ includes $range(P(u,v), z_{i+1})$, then statement (ii) trivially follows. Otherwise, suppose that the slope of (z_{i+1}, z''_{i+1}) is contained in $range(P(u,v), z_{i+1})$, as in Fig. 5(a). Since the smallest range containing $range(P(u,v), z_i)$ and (z_i, z'_i) has an angle smaller than π, and since the slope of (z_{i+1}, z'_{i+1}) is between the slope of (z_i, z'_i) and the slope of (z_{i+1}, z''_{i+1}), statement (ii) follows.
- $*$ Suppose that $range(z_{i+1})$ has an angle larger than or equal to π (see Fig. 5(b)). Then, since $range(P(u,v), z_{i+1})$ has an angle smaller than π and since both the smallest range containing $range(P(u,v), z_i)$ and (z_i, z'_i) and the smallest range containing $range(P(u,v), z_i)$ and (z_i, z''_i) have angles smaller than π, it follows that $range(P(u,v), z_{i+1})$ is a subrange of $range(z_{i+1})$. Further, since the slope of (z_{i+1}, z'_{i+1}) is between the slope of (z_i, z'_i) and the slopes of $range(P(u,v), z_{i+1})$, the smallest range containing $range(P(u,v), z_{i+1})$ and (z_{i+1}, z'_{i+1}) has an angle smaller than π. Analogously, it is possible to prove that the smallest range containing $range(P(u,v), z_{i+1})$ and (z_{i+1}, z''_{i+1}) has an angle smaller than π.

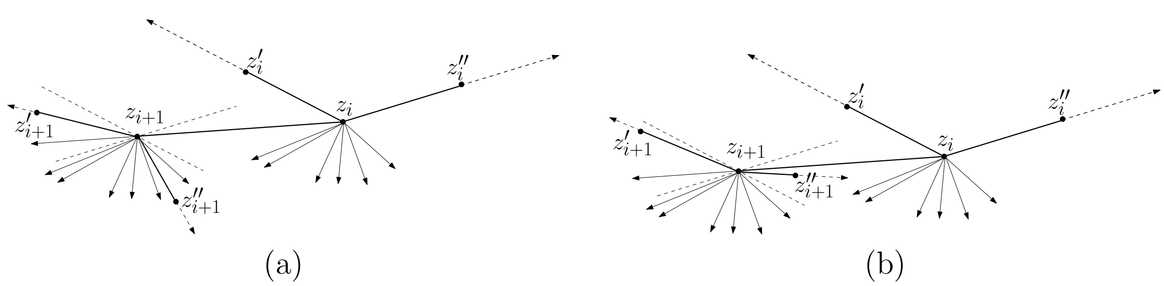

Figure 6: $range(z_i)$ has an angle larger than or equal to π, edge (z_i, z_{i+1}) has slope between the one of (z_i, z'_i) and the ones of $range(P(u,v), z_i)$, and (a) $range(z_{i+1})$ has an angle smaller than π or (b) $range(z_{i+1})$ has an angle larger than or equal to π.

- The case in which (z_i, z_{i+1}) has slope between the one of (z_i, z'_i) and the ones of $range(P(u,v), z_i)$ (see Fig. 6(a)–(b)) and the case in which (z_i, z_{i+1}) has slope between the ones of $range(P(u,v), z_i)$ and the one of (z_i, z''_i) can be discussed analogously to the case in which (z_i, z_{i+1}) is inside $range(P(u,v), z_i)$.

- The case in which $range(z_i)$ has an angle smaller than π can be discussed analogously and in a simpler way than the case in which $range(z_i)$ has an angle larger than or equal to π.

□

A simple modification of the algorithm presented in [4] constructs strictly convex drawings of trees. Hence, monotone drawings exist for all trees.

We introduce *slope-disjoint drawings* of trees and show that they are monotone. Let T be a tree rooted at a node r. Denote by $T(u)$ the subtree of T rooted at a node u. A *slope-disjoint* drawing of T is such that: (P1) For every node $u \in T$, there exist two angles $\alpha_1(u)$ and $\alpha_2(u)$, with $0 < \alpha_1(u) < \alpha_2(u) < \pi$, such that, for every edge e that is either in $T(u)$ or that connects u with its parent, it holds that $\alpha_1(u) < slope(e) < \alpha_2(u)$; (P2) for every two nodes $u, v \in T$ with v child of u, it holds that $\alpha_1(u) < \alpha_1(v) < \alpha_2(v) < \alpha_2(u)$; (P3) for every two nodes v_1, v_2 with the same parent, it holds that $\alpha_1(v_1) < \alpha_2(v_1) < \alpha_1(v_2) < \alpha_2(v_2)$. We have the following:

Theorem 1 *Every slope-disjoint drawing of a tree is monotone.*

Proof: Let T be a tree and let Γ be a slope-disjoint drawing of T. We show that, for every two vertices $u, v \in T$, a monotone path between u and v exists in Γ. Let w be the lowest common ancestor of u and v in T.

If $w = u$, then, by Property P1, for every edge e in the path $P(u, v)$, $0 < slope(e) < \pi$. Hence, $P(u, v)$ is monotone with respect to a half-line with slope $\pi/2$. Analogously, if $w = v$ then $P(u, v)$ is monotone with respect to a half-line with slope $-\pi/2$. If $w \neq u, v$, let u' and v' be the children of w in T such that $u \in T(u')$ and $v \in T(v')$. Path $P(u, v)$ is composed of path $P(u, w)$ and of path $P(w, v)$. As before, $P(u, w)$ is monotone with respect to a half-line with slope $-\pi/2$. By Property P1, for every edge $e \in P(u, w)$, $\alpha_1(u') - \pi < slope(e) < \alpha_2(u') - \pi$. Hence, $\alpha_1(u') < slope(l) < \alpha_2(u')$, for each half-line l contained into the closed wedge $opp(P(u, w))$. Analogously, $P(w, v)$ is monotone with respect to a half-line with slope $\pi/2$ and, by Property P1, for every edge $e \in P(w, v)$, $\alpha_1(v') < slope(e) < \alpha_2(v')$. Hence, $\alpha_1(v') < slope(l) < \alpha_2(v')$, for each half-line l contained into the closed wedge $range(P(w, v))$. Finally, since u' and v' are children of the same node, by Property P3 $\alpha_1(u') < \alpha_2(u') < \alpha_1(v') < \alpha_2(v')$ (the case in which $\alpha_1(v') < \alpha_2(v') < \alpha_1(u') < \alpha_2(u')$ being symmetric). Since $\alpha_1(u') < slope(l) < \alpha_2(u')$ for each half-line l contained into the closed wedge $opp(P(u, w))$ and since $\alpha_1(v') < slope(l) < \alpha_2(v')$ for each half-line l contained into the closed wedge $range(P(w, v))$, we have $opp(P(u, w)) \cap range(P(w, v)) = \emptyset$. By Corollary 1, $P(u, v)$ is monotone.

□

By Theorem 1, as long as the slopes of the edges in a drawing of a tree T guarantee the slope-disjoint property, one can *arbitrarily* assign lengths to such edges always obtaining a monotone drawing of T. In the following we present two algorithms for constructing slope-disjoint drawings of any tree T. In both algorithms, we individuate a suitable set of elements of the Stern-Brocot tree $\mathcal{SB}$. Each of such elements, say $s = y/x$, is then used as a slope of an edge of T in the drawing.

Algorithm BFS-based: Consider the first $\lceil \log_2(n) \rceil$ levels of the Stern-Brocot tree $\mathcal{SB}$. Such levels contain a total number of at least $n - 1$ elements y/x of $\mathcal{SB}$. Order such elements by increasing value of the ratio y/x and consider the first $n-1$ elements in such an order S, say $s_1 = y_1/x_1, s_2 = y_2/x_2, \ldots, s_{n-1} = y_{n-1}/x_{n-1}$. Consider the subtrees of r, say $T_1(r), T_2(r), \ldots, T_{k(r)}(r)$. Assign to $T_i(r)$ the $|T_i(r)|$ elements of S from the $(1 + \sum_{j=1}^{i-1} |T_j(r)|)$-th to the $(\sum_{j=1}^{i} |T_j(r)|)$-th. Consider a node u of

T and suppose that a sub-sequence $S(u) = s_a, s_{a+1}, \ldots, s_b$ of S has been assigned to $T(u)$, where $|T(u)| = b - a$. Consider the subtrees $T_1(u), T_2(u), \ldots, T_{k(u)}(u)$ of u and assign to $T_i(u)$ the $|T_i(u)|$ elements of $S(u)$ from the $(1 + \sum_{j=1}^{i-1} |T_j(u)|)$-th to the $(\sum_{j=1}^{i} |T_j(u)|)$-th. Now we construct a grid drawing of T. Place r at $(0, 0)$. Consider a node u of T, suppose that a sequence $S(u) = s_a, s_{a+1}, \ldots, s_b$ of S has been assigned to $T(u)$ and suppose that the parent $p(u)$ of u has been already placed at the grid point $(p_x(u), p_y(u))$. Place u at grid point $(p_x(u) + x_b, p_y(u) + y_b)$, where $s_b = y_b/x_b$. See Fig. 7 for an example of application. We have the following:

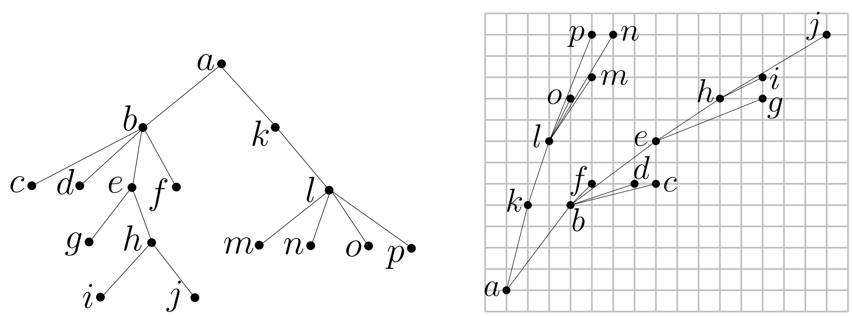

Figure 7: (a) A tree T. (b) The drawing of T constructed by Algorithm BFS-based.

Theorem 2 *Let T be a tree. Then, Algorithm BFS-based constructs a monotone drawing of T on a grid of area $O(n^{1.6}) \times O(n^{1.6})$.*

Proof: We prove that the drawing Γ constructed by Algorithm BFS-based is slope-disjoint and hence monotone. In order to this, we describe how to choose values $\alpha_1(u)$ and $\alpha_2(u)$ for every node $u \in T$. Recall that a sequence $S(u)$ is associated to each node $u \in T$ in such a way that the edge connecting u to its parent and all the edges in $T(u)$ have slopes in $S(u)$. Choose as $\alpha_1(r)$ any angle larger than $0°$ and smaller than the smallest angle $\arctan \frac{y}{x}$, for all elements y/x in $S(r)$. Analogously, choose as $\alpha_2(r)$ any angle smaller than $90°$ and larger than the largest angle $\arctan \frac{y}{x}$, for all elements y/x in $S(r)$. Now suppose that values $\alpha_1(u)$ and $\alpha_2(u)$ have been set for a node $u \in T$ and not for a child v of u. Choose as $\alpha_1(v)$ any angle: (i) smaller than the smallest angle $\arctan \frac{y}{x}$, for all elements y/x in $S(v)$, (ii) larger than $\alpha_1(u)$, (iii) larger than the largest angle $\arctan \frac{y_1}{x_1}$, for all elements y_1/x_1 not in $S(v)$ such that $y_1/x_1 < y_2/x_2$, for some element y_2/x_2 in $S(v)$, and (iv) larger than the largest value $\alpha_2(z)$, where z is a child of u such that $\alpha_2(z)$ has been already set and $y_1/x_1 < y_2/x_2$, for some element y_2/x_2 in $S(v)$ and some element y_1/x_1 in $S(z)$. Set $\alpha_2(v)$ in a symmetric way. It is easy to see that Properties P1–P3 are satisfied by the assignment of values $\alpha_1(u)$ and $\alpha_2(u)$ for every node $u \in T$.

It remains to show that Γ lies on a grid area $O(n^{1,6}) \times O(n^{1,6})$. Consider the node z of T which has greatest x-coordinate and consider the path $P(z, r)$ from z to r. An edge (u, v) of such a path, where u is the parent of v, has x-extension equal to x_1, where

y_1/x_1 is an element of $\mathcal{SB}$. Hence, the x-extension of $P(z,r)$ and the x-extension of Γ are bounded by the sum of all the x-coordinates of the elements of $\mathcal{SB}$. By Property 1 and by $\sum_{j=0}^{i-1} 3^j < 3^i$, the x-extension of Γ is $O(3^{\log_2 n}) = O(n^{1/\log_3 2}) = O(n^{1.6})$. Analogously, the y-extension of Γ is $O(n^{1.6})$ and the theorem follows. □

Algorithm DFS-based: Consider the sequence S composed of the first $n-1$ elements $1/1, 2/1, \ldots, n-1/1$ of the rightmost path of $\mathcal{SB}$. Assign sub-sequences of S to the subtrees of T and construct a grid drawing in the same way as in Algorithm BFS-based. We have the following.

Theorem 3 *Let T be a tree. Then, Algorithm DFS-based constructs a monotone drawing of T on a grid of area $O(n^2) \times O(n)$.*

Proof: The proof that the drawing Γ constructed by Algorithm DFS-based is slope-disjoint and hence monotone is the same as for Algorithm BFS-based.

We show that Γ lies on a grid area $O(n^2) \times O(n)$. Every edge has x-extension equal to 1, hence the width of Γ is $O(n)$. The y-extension of an edge is at most $n-1$, hence the height of Γ is $O(n^2)$. □

As a further consequence of Theorem 1, we have the following:

Corollary 2 *Every (even non-planar) graph admits a monotone drawing.*

Namely, for any graph G, construct a monotone drawing of a spanning tree T of G with vertices in general position. Draw the other edges of G as segments, obtaining a straight-line drawing of G in which, for any pair of vertices, there exists a monotone path (the one whose edges belong to T). Such a drawing is a monotone drawing of G.

5 Planar Monotone Drawings of Biconnected Graphs

First, we restate, using the terminology of this paper, the well-known result of [5].

Lemma 6 *[5] Let G be a biconnected planar graph with a given planar embedding such that each split pair u, v is incident to the outer face and each maximal split component of u, v has at least one edge incident to the outer face but, possibly, for edge (u, v). Then, G admits a strictly convex drawing with the given embedding in which the outer face is drawn as an arbitrary strictly convex polygon.*

Let Γ be a monotone drawing, d any direction, and k a positive value. A *directional-scale*, denoted by $\mathcal{DS}(d, k)$, is an affine transformation defined as follows. Rotate Γ by an angle δ until d is orthogonal to the x-axis. Scale Γ by $(1, k)$ (i.e., multiply its y-coordinates by k). Rotate back the obtained drawing by an angle $-\delta$.

Lemma 7 *Let Γ be a monotone drawing and d a direction such that no edge in Γ is parallel to d. For any $\alpha > 0$ a directional-scale $\mathcal{DS}(d - \frac{\pi}{2}, k(\alpha))$ exists that transforms Γ into a monotone drawing in which the slope of any edge is between $d - \alpha$ and $d + \alpha$.*

Proof: First observe that, when Γ is rotated of an angle δ until $d - \frac{\pi}{2}$ is orthogonal to the x-axis, when it is scaled, and when it is rotated back of an angle δ, by Lemma 5, Γ remains monotone.

Consider any edge $(v, w) \in \Gamma$. As (v, w) is not parallel to d, there exists an angle $\beta < \frac{\pi}{2}$ such that $d - \beta < slope(v, w) < d + \beta$. Further, $\tan(\beta_P) = \frac{\Delta_y}{\Delta_x}$, where $\Delta_y = y(w) - y(v)$ and $\Delta_x = x(w) - x(v)$.

Observe that scaling Γ by $(1, k)$, with $k < 1$, reduces Δ_y by a factor of $\frac{1}{k}$, while it keeps Δ_x unchanged. Hence, $\tan(\beta)$ is reduced by a factor of $\frac{1}{k}$, which implies that β is reduced by a linear factor proportional to $\frac{1}{k}$. As $k \to 0$ implies $\beta \to 0$, it follows that for every $\alpha > 0$ there exists a $k > 0$ such that $\beta < \alpha$. □

A path monotone with respect to a direction d is (α, d)-*monotone* if, for each edge e, $d - \alpha < slope(e) < d + \alpha$. A path from a vertex u to a vertex v is an (α, d_1, d_2)-*path* if it is a composition of a (α, d_1)-monotone path from u to a vertex w and of a (α, d_2)-monotone path from w to v. Let p_N, p_S, p_W, and p_E be four points in the plane such that p_W is inside triangle $\triangle(p_N, p_S, p_E)$, $\widehat{p_W p_S p_E} = \widehat{p_W p_N p_E}$, and $\widehat{p_W p_S p_N} + 2\widehat{p_W p_S p_E} < \frac{\pi}{2}$. Quadrilateral (p_N, p_E, p_S, p_W) is a *boomerang* (see Fig. 8).

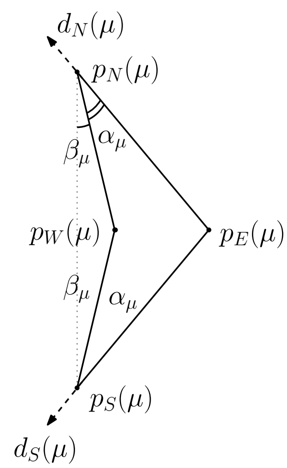

Figure 8: A boomerang.

Let G be a biconnected graph and $\mathcal{T}$ be the SPQR-decomposition of G rooted at an edge e. We prove that G admits a planar monotone drawing by means of an inductive algorithm which, given a component μ of $\mathcal{T}$ with poles u and v, and a boomerang $boom(\mu) = (p_N(\mu), p_E(\mu), p_S(\mu), p_W(\mu))$, constructs a drawing Γ_μ of $pert(\mu)$ satisfying the following properties. Let $d_N(\mu)$ be the half-line starting at $p_E(\mu)$ through $p_N(\mu)$, let $d_S(\mu)$ be the half-line starting at $p_E(\mu)$ through $p_S(\mu)$, let α_μ be $p_W(\mu)\widehat{p_S(\mu)}p_E(\mu) = p_W(\mu)\widehat{p_N(\mu)}p_E(\mu)$, and let $\beta_\mu = p_W(\mu)\widehat{p_S(\mu)}p_N(\mu)$. (A) Γ_μ is monotone; (B) with the possible exception of edge (u, v), Γ_μ is contained into $boom(\mu)$, with u drawn on $p_N(\mu)$ and v on $p_S(\mu)$; (C) each vertex $w \in pert(\mu)$ belongs to a $(\alpha_\mu, -d_N(\mu), d_S(\mu))$-path from u to v. Observe that Property C implies that in Γ_μ there exists a path between the poles that is monotone with respect to the line through them and that Property B implies the planarity of Γ_μ.

Lemma 8 *Let μ be a component of $\mathcal{T}$. Every $(\alpha_\mu, -d_N(\mu), d_S(\mu))$-path from u to v is monotone with respect to the half-line from u through v.*

Proof: The proof is based on the fact that $\beta_\mu + 2\alpha_\mu < \frac{\pi}{2}$. Hence, the direction of any edge of the $(\alpha_\mu, -d_N(\mu))$-monotone path is contained in the fourth quadrant, while the direction of any edge of the $(\alpha_\mu, d_S(\mu))$-monotone path is contained in the third quadrant. It follows that the range of any $(\alpha_\mu, -d_N(\mu), d_S(\mu))$-path $P(u,v)$ from u to v contains the direction defined by the half-line from u through v. □

Let $\mu_1, \dots, \mu_k$ be the children of μ in $\mathcal{T}$, with poles $(u_1, v_1), \dots, (u_k, v_k)$. We construct a drawing Γ_μ satisfying Properties A–C by composing drawings $\Gamma_{\mu_1}, \dots, \Gamma_{\mu_k}$, which are constructed inductively, as follows.

If μ is a Q-node, then draw an edge between $p_N(\mu)$ and $p_S(\mu)$.

If μ is an S-node (see Fig. 9(a)), then let p be the intersection point between segment $\overline{p_W(\mu)p_E(\mu)}$ and the bisector line of $p_W(\mu)\widehat{p_N(\mu)}p_E(\mu)$. Consider k equidistant points $p_1, \dots, p_k$ on segment $\overline{p_N(\mu)p}$ such that $p_1 = p_N(\mu)$ and $p_k = p$. For each μ_i, with $i = 1, \dots, k-1$, consider a boomerang $boom(\mu_i) = (p_N(\mu_i),\ p_E(\mu_i),\ p_S(\mu_i),\ p_W(\mu_i))$ such that $p_N(\mu_i) = p_i$, $p_S(\mu_i) = p_{i+1}$, and $p_E(\mu_i)$ and $p_W(\mu_i)$ determine $\beta_{\mu_i} + 2\alpha_{\mu_i} < \frac{\alpha_\mu}{2}$. Apply the inductive algorithm to μ_i and $boom(\mu_i)$. Also, consider a boomerang $boom(\mu_k) = (p_N(\mu_k),\ p_E(\mu_k),\ p_S(\mu_k),\ p_W(\mu_k))$ such that $p_N(\mu_k) = p$, $p_S(\mu_k) = p_S(\mu)$, and $p_E(\mu_k)$ and $p_W(\mu_k)$ determine $\beta_{\mu_k} + 2\alpha_{\mu_k} < \frac{\alpha_\mu}{2}$. Apply the inductive algorithm to μ_k and $boom(\mu_k)$.

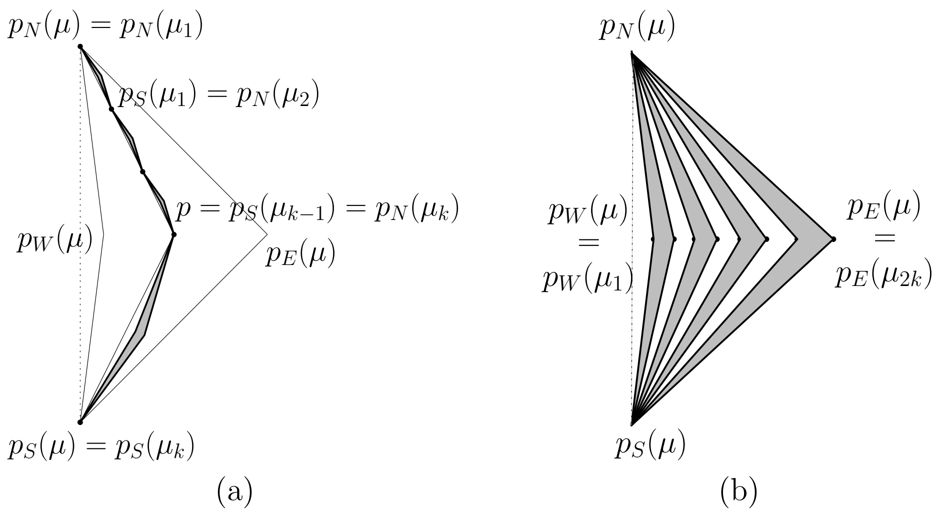

Figure 9: The construction rules for an S-node (a) and for a P-node (b).

If μ is a P-node (see Fig. 9(b)), then consider $2k$ points $p_1, \dots, p_{2k}$ on segment $\overline{p_W(\mu)p_E(\mu)}$ such that $p_1 = p_W(\mu)$, $p_{2k} = p_E(\mu)$, and $p_i\widehat{p_N(\mu)}p_{i+1} = \frac{\alpha_\mu}{2k-1}$, for each

$i = 1, \dots, 2k-1$. For each μ_i, with $i = 1, \dots, k$, consider a boomerang $boom(\mu_i) = (p_N(\mu_i), p_E(\mu_i), p_S(\mu_i), p_W(\mu_i))$ such that $p_N(\mu_i) = p_N(\mu)$, $p_S(\mu_i) = p_S(\mu)$, $p_W(\mu_i) = p_{2i-1}$, and $p_E(\mu_i) = p_{2i}$. Apply the inductive algorithm to μ_i and $boom(\mu_i)$.

If μ is an R-node, then consider the graph G' obtained by removing v and its incident edges from $skel(\mu)$. Since $skel(\mu)$ is triconnected, G' is a biconnected graph whose possible split pairs are incident to the outer face. Further, each of such split pairs separates at most three maximal split components, and in this case one of them is an edge. By Lemma 6, G' admits a convex drawing whose outer face is represented by any strictly convex polygon. Consider a strictly convex polygon C with one vertex placed on p_N, one vertex placed on p_E, and $m-2$ vertices placed inside $boom(\mu)$ so that they are visible from $p_S(\mu)$ inside $boom(\mu)$ and the internal angle incident to $p_E(\mu)$ is smaller than $\frac{\pi}{2}$ (see Fig. 10(a)). Construct a convex drawing $\Gamma(G')$ of G' such that the vertices of the outer face of G' are placed on the vertices of C, with u placed on $p_N(\mu)$. By Lemma 2, $\Gamma(G')$ is monotone. Slightly perturb the position of the vertices of $\Gamma(G')$ so that no two parallel edges exist and no edge is orthogonal to $d_N(\mu)$. Apply a directional-scale $\mathcal{DS}(d_N(\mu) - \frac{\pi}{2}, k(\frac{\alpha_\mu}{2}))$ to $\Gamma(G')$. By Lemma 7, for every edge $e \in G'$, $slope(-d_N(\mu)) - \frac{\alpha_\mu}{2} < slope(e) < slope(-d_N(\mu)) + \frac{\alpha_\mu}{2})$. Further, by Lemma 4 and by the fact that the internal angle of C incident to $p_N(\mu)$ is smaller than $\frac{\alpha_\mu}{2} < \frac{\pi}{2}$, for every vertex $w \in G'$, a $(\frac{\alpha_\mu}{2}, -d_N(\mu))$-monotone path exists from u to w. Let $\Gamma(skel(\mu))$ be the drawing of $skel(\mu)$ obtained from $\Gamma(G')$ by placing v on $p_S(\mu)$ and drawing its incident edges (see Fig. 10(b)). We have the following:

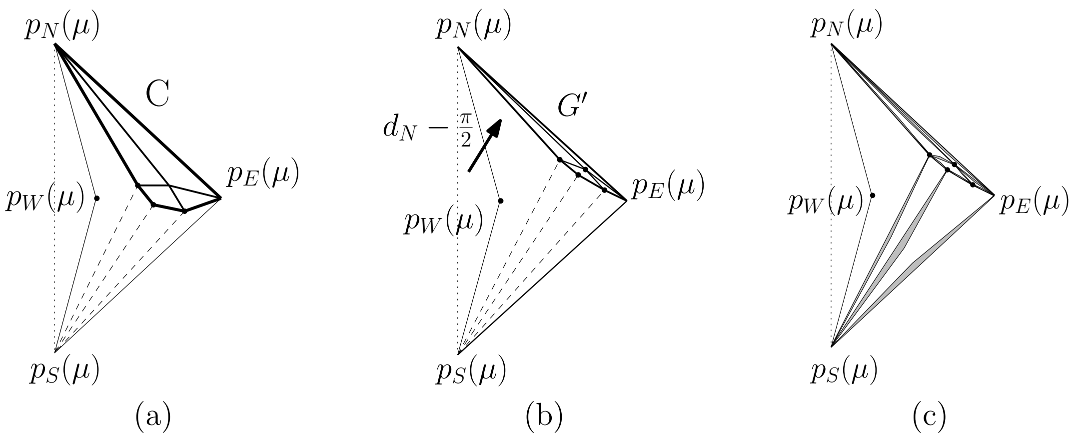

Figure 10: Two phases of the construction for an R-node: (a) definition of the strictly convex polygon C and (b) the directional-scale applied to G'.

Claim 1 $\Gamma(skel(\mu))$ *is monotone.*

Proof: Every two vertices different from v are connected in $\Gamma(skel(\mu))$ by the same monotone path as in $\Gamma(G')$. Also, for every vertex $w \in skel(\mu)$, consider a path $P(u, v)$ from u to v that is a composition of a $(\frac{\alpha_\mu}{2}, -d_N(\mu))$-monotone path from u to a vertex w'

adjacent to v passing through w and of the $(\alpha_\mu, d_S(\mu))$-monotone edge (w', v). Observe that $P(u,v)$ is an $(\alpha_\mu, -d_N(\mu), d_S(\mu))$-path. Since, by Lemma 8, $P(u,v)$ is monotone, the subpath of $P(u,v)$ between w and v is monotone, by Property 2. □

Consider a drawing $\Gamma'(skel(\mu))$ of a subdivision of $skel(\mu)$ obtained as a subdivision of $\Gamma(skel(\mu))$. We have the following:

Claim 2 *$\Gamma'(skel(\mu))$ is monotone.*

Proof: The drawing obtained by adding edge (u,v) to $\Gamma(skel(\mu))$ is convex and has no parallel edge. Hence, any subdivision Γ' of $\Gamma(skel(\mu))$ is monotone, by Lemma 3. We claim that for each pair of vertices there exists a monotone path not containing (u,v). The existence of a monotone path between any two vertices of $skel(\mu)$ not containing (u,v) derives from the fact that $\Gamma(skel(\mu))$ is monotone. The existence of a monotone path between any two vertices both different from u and v not containing (u,v) derives from the fact that in this case every path containing (u,v) has two consecutive angles smaller than $\frac{\pi}{2}$. The only case that remains is when one of the vertices is a pole, say u, and the other is a subdivision vertex w. Let (w_1, w_2) be the edge of $skel(\mu)$ whose subdivision vertex is w. Assume, without loss of generality, that (w_1, w_2) has a negative projection on $d_N(\mu)$. Consider the path $P(u,v)$ from u to v that is a composition of the monotone path $P(u, w_1)$ between u and w_1, of the monotone path (w_1, w', w_2), and of the monotone path from w_2 to v. As $d_N(\mu) - \alpha_\mu < slope(w_1, w') < d_N(\mu) + \alpha_\mu$, $P(u,v)$ is monotone, and hence the subpath of $P(u,v)$ between u and w' is monotone, by Property 2. □

Consider the pair of vertices x, y belonging to the subdivision of $skel(\mu)$ such that the range $range(P(x,y))$ of the monotone path $P(x,y)$ between them in $\Gamma'(skel(\mu))$ creates the largest angle $\angle(x,y)$ among all the pairs of vertices. Let $\gamma = \pi - \angle(x,y)$. Let δ be the smallest angle between two adjacent edges in $\Gamma(skel(\mu))$. For each μ_i, with $i = 1, \dots, k$, let $p_N(\mu_i)$ and $p_S(\mu_i)$ be the points where u_i and v_i have been drawn in $\Gamma(skel(\mu))$, respectively. Consider a boomerang $boom(\mu_i) = (p_N(\mu_i), p_E(\mu_i), p_S(\mu_i), p_W(\mu_i))$ such that $p_E(\mu_i)$ and $p_W(\mu_i)$ determine $\beta_{\mu_i} + 2\alpha_{\mu_i} < \min\{\frac{\delta}{2}, \frac{\gamma}{2}\}$. For each μ_p such that either $p_N(\mu_p)$ and $p_S(\mu_p)$ lie on the vertices of C or $p_S(\mu_p) = p_S(\mu)$, choose points $p_W(\mu_p)$ and $p_E(\mu_p)$ inside $boom(\mu)$. Then, apply the inductive algorithm to μ_i, with poles u_i and v_i, and $boom(\mu_i)$ (see Fig. 10(c)).

In the following we prove that the above described algorithm constructs a planar monotone drawing of every biconnected planar graph.

Theorem 4 *Every biconnected planar graph admits a planar monotone drawing.*

Proof: Let $\mathcal{T}$ be the SPQR-decomposition of a biconnected graph G, rooted at any Q-node μ_e corresponding to an edge e. Consider a boomerang $boom(\mu_e) = (p_N(\mu_e), p_E(\mu_e), p_S(\mu_e), p_W(\mu_e))$ such that $x(p_N(\mu_e)) = x(p_S(\mu_e)) < x(p_W(\mu_e)) < x(p_E(\mu_e))$, $y(p_S(\mu_e)) < y(p_W(\mu_e)) = y(p_E(\mu_e)) < y(p_N(\mu_e))$, and $\beta_{\mu_e} + 2\alpha_{\mu_e} < \frac{\pi}{2}$. Apply the inductive algorithm described above to μ_e and $boom(\mu_e)$. We prove that the resulting

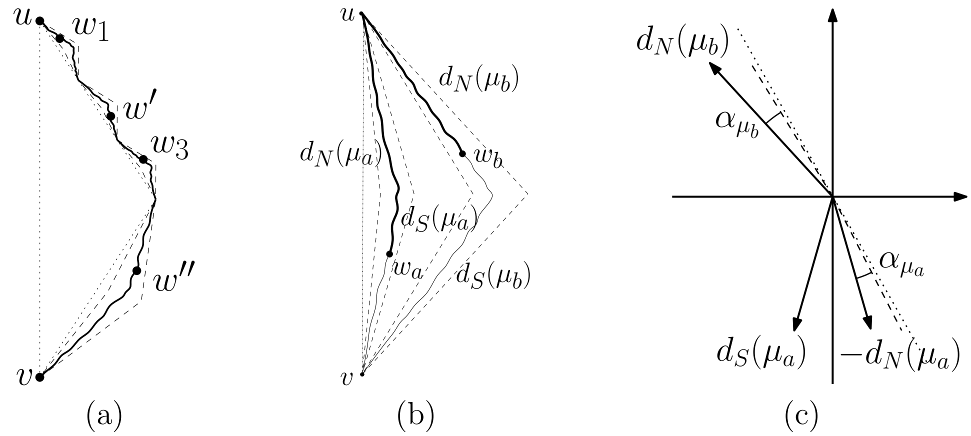

Figure 11: Γ_μ satisfies Property A (a) if μ is an S-node and (b)–(c) if μ is a P-node.

drawing is monotone by showing that at each step of the induction the constructed drawing satisfies Properties A–C. This is trivial if μ is a Q-node. Otherwise, μ is an S-node, a P-node, or an R-node and the statement is proved by the following claims:

Claim 3 *If μ is an S-node, Γ_μ satisfies Property A.*

Proof: Refer to Fig. 11(a). Consider any two vertices $w', w'' \in pert(\mu)$ and the components μ_a and μ_b such that $w' \in pert(\mu_a)$ and $w'' \in pert(\mu_b)$. If $a = b$, then a monotone path between w' and w'' exists by induction. Otherwise, for each μ_i, consider a vertex w_i, where $w_a = w'$ and $w_b = w''$. For each μ_i, with $i = 1, \dots, k$, consider a $(\alpha_{\mu_i}, -d_N(\mu_i), d_S(\mu_i))$-path $P(u_i, v_i)$ from u_i to v_i containing w_i. Observe that such paths exist since, for each μ_i, Γ_{μ_i} satisfies Property C. Consider a path $P(u_i, v_i)$ with $1 \leq i \leq k-1$. Since $\beta_{\mu_i} + 2\alpha_{\mu_i} < \frac{\alpha_\mu}{2}$, and since $p_N(\mu_i)$ and $p_S(\mu_i)$ lie on the bisector line of α_μ, for each edge $e \in P(u_i, v_i)$, it holds $slope(e) < \beta_\mu + \frac{\alpha_\mu}{2} + \beta_{\mu_i} + 2\alpha_{\mu_i} < \beta_\mu + \alpha_\mu < \beta_\mu + 2\alpha_\mu = d_N(\mu) + \alpha$, and $slope(e) > \beta_\mu + \frac{\alpha_\mu}{2} - (\beta_{\mu_i} + 2\alpha_{\mu_i}) > \beta_\mu = d_N(\mu) - \alpha_\mu$. Hence, $P(u_i, v_i)$ is $(\alpha_\mu, -d_N(\mu))$-monotone. Analogously, $P(u_k, v_k)$ is $(\alpha_\mu, d_S(\mu))$-monotone. Therefore, the path $P(u, v)$ composed of all the paths $P(u_i, v_i)$ is an $(\alpha_\mu, -d_N(\mu), d_S(\mu))$-path. By Lemma 8, $P(u, v)$ is monotone. Hence, by Property 2, the subpath of $P(u, v)$ between w' and w'' is monotone, as well, and Γ_μ satisfies Property A. □

Claim 4 *If μ is an S-node, Γ_μ satisfies Properties B and C.*

Proof: We prove that Γ_μ satisfies Property B. Observe that, for each component $\mu_1, \dots, \mu_{k-1}$, $boom(\mu_i)$ is such that $p_N(\mu_i)$ and $p_S(\mu_i)$ lie on segment $\overline{p_N(\mu)p}$, which creates angles equal to $\frac{\alpha_\mu}{2}$ with $\overline{p_N(\mu)p_W(\mu)}$ and with $\overline{p_N(\mu)p_E(\mu)}$. Further, $\beta_{\mu_i} + \alpha_{\mu_i} < \beta_{\mu_i} + 2\alpha_{\mu_i} < \frac{\alpha_\mu}{2}$. Hence, $boom(\mu_i)$ is contained inside $boom(\mu)$. Analogously, it is possible to show that $boom(\mu_k)$ is contained inside $boom(\mu)$. As, by induction, $pert(\mu_i)$ is contained into $boom(\mu_i)$, for each $i = 1, \dots, k$, Property B holds for Γ_μ.

We prove that Γ_μ satisfies Property C. Observe that, for every vertex $w \in pert(\mu)$, it is possible to choose any vertex $w' \in pert(\mu)$ such that w and w' do not belong to the same component μ_i and to show that there exists a $(\alpha_\mu, -d_N(\mu), d_S(\mu))$-path $P(u,v)$ from u to v passing through w and w' with the same argument as in Claim 3. □

Claim 5 *If μ is a P-node, Γ_μ satisfies Property A.*

Proof: Consider any two vertices $w_a, w_b \in pert(\mu)$ and the components μ_a and μ_b such that $w_a \in pert(\mu_a)$ and $w_b \in pert(\mu_b)$. If $a = b$, then a monotone path between w_a and w_b exists by induction. Otherwise, consider the $(\alpha_{\mu_a}, -d_N(\mu_a), d_S(\mu_a))$-path $P_a(u,v)$ from u to v through w_a and the $(\alpha_{\mu_b}, d_N(\mu_b), -d_S(\mu_b))$-path $P_b(v,u)$ from v to u through w_b, which exist by induction (Property C). Suppose that w_b lies on the $(\alpha_{\mu_b}, d_N(\mu_b))$-monotone path $P(w_b,u)$ from w_b to u that is a subpath of $P_b(v,u)$, the other case being analogous. Consider the $(\alpha_{\mu_a}, -d_N(\mu_a), d_S(\mu_a))$-path $P(u,w_a)$ that is a subpath of $P_a(u,v)$. We show that the path $P(w_b,w_a)$ composed of $P(w_b,u)$ and $P(u,w_a)$ is monotone. When translated to the origin of the axes, $d_N(\mu_b)$ is in the second quadrant, $-d_N(\mu_a)$ in the fourth quadrant, and $d_S(\mu_a)$ in the third quadrant. By construction, the wedge delimited by $d_N(\mu_b)$ and $-d_N(\mu_a)$ and containing the third quadrant has an angle smaller than or equal to $\pi - 2\frac{\alpha_\mu}{2k-1}$. Since, by definition, every edge of $P(w_b,u)$ creates an angle with $d_N(\mu_b)$ that is smaller than $\alpha_{\mu_b} = \frac{\alpha_\mu}{2k-1}$ and every edge of $P(u,w_a)$ creates an angle with $-d_N(\mu_a)$ that is smaller than $\alpha_{\mu_a} = \frac{\alpha_\mu}{2k-1}$, it follows that the slopes of all the edges of $P(w_b,w_a)$ lie inside a wedge having an angle smaller than π. Hence, $P(w_b,w_a)$ is monotone. □

Claim 6 *If μ is a P-node, Γ_μ satisfies Properties B and C.*

Proof: We prove that Γ_μ satisfies Property B. Each boomerang $boom(\mu_i)$ is contained inside $boom(\mu)$, by construction, and each vertex $w \in pert(\mu_i)$ is contained inside $boom(\mu_i)$, by induction (Property B).

We prove that Γ_μ satisfies Property C. It is sufficient to observe that, for each vertex $w \in pert(\mu)$, the $(\alpha_\mu, -d_N(\mu), d_S(\mu))$-path passing through w coincides with the $(\alpha_{\mu_i}, -d_N(\mu_i), d_S(\mu_i))$-path passing through w, where μ_i is the component such that $w \in pert(\mu_i)$, which exists by induction (Property C). □

Claim 7 *If μ is an R-node, Γ_μ satisfies Property A.*

Proof: Consider any two vertices w_a and w_b of $pert(\mu)$ and the components μ_a and μ_b such that $w_a \in pert(\mu_a)$ and $w_b \in pert(\mu_b)$. Let e_a and e_b be the virtual edges of $skel(\mu)$ corresponding to μ_a and μ_b, respectively. If $a = b$ by induction there trivially exists a monotone path between w_a and w_b. Otherwise, consider the monotone drawing $\Gamma'(skel(\mu))$ of a subdivision of $skel(\mu)$ and the monotone path $P_{a,b}$ between the subdivision vertex of e_a and the subdivision vertex of e_b. By construction, $\pi - range(P_{a,b}) \geq \gamma$. As $\beta_{\mu_i} + 2\alpha_{\mu_i} < \min\{\frac{\delta}{2}, \frac{\gamma}{2}\} \leq \frac{\gamma}{2}$, for the path $P(w_a, w_b)$ that is obtained by replacing each edge e_i of $P_{a,b}$ with the corresponding path of $pert(\mu_{e_i})$ it holds that $range(P(w_a, w_b)) < \pi$. □

Claim 8 *If μ is an R-node, Γ_μ satisfies Properties B and C.*

Proof: We prove that Γ_μ satisfies Property B. For each component μ_p whose corresponding virtual edge is incident to the outer face of $skel(\mu)$, points $p_W(\mu_p)$ and $p_E(\mu_p)$ are inside $boom(\mu)$, by construction. Further, as for each component μ_i it holds $\beta_{mu_i} + 2\alpha_{\mu_i} < \min\{\frac{\delta}{2}, \frac{\gamma}{2}\} \leq \frac{\delta}{2}$, there is no intersection between any two $boom(\mu_p)$ and $boom(\mu_q)$, with $p \neq q$. Hence, all such boomerangs are contained into $boom(\mu)$. As, by induction, $pert(\mu_i)$ is contained into $boom(\mu_i)$, for each $i = 1, \ldots, k$, Property B holds for Γ_μ.

We prove that Γ_μ satisfies Property C. Let $\Gamma(G')$ be drawing of the subgraph of $skel(\mu)$ induced by the vertices of $skel(\mu) \setminus \{v\}$. Observe that, as the position of the vertices in $\Gamma(G')$ has been perturbed before applying the directional-scale $\mathcal{DS}(d_N(\mu) - \frac{\pi}{2}, k(\frac{\alpha_\mu}{2}))$, every edge e in $\Gamma(G')$ is such that $d_N(\mu) - \frac{\alpha_\mu}{2}) < slope(e) < d_N(\mu) + \frac{\alpha_\mu}{2})$.

Consider any vertex $w \in pert(\mu)$ and let μ_p be the child component such that $w \in pert(\mu_p)$. Let (u_p, v_p) be the virtual edge in $\Gamma(skel(\mu))$ corresponding to μ_p. Two cases are possible: either (u_p, v_p) is adjacent to v or not. In the first case, consider the path $P(u, v)$ from u to v that is a composition of the $(\frac{\alpha_\mu}{2}), -d_N(\mu))$-monotone path $P(u, u_p)$ between u and u_p and of the $(\alpha_{\mu_p}, -d_N(\mu_p), d_S(\mu_p))$-path $P(u_p, v_p)$ from u_p to $v_p = v$ passing through w, which exists by induction. As $\beta_{\mu_i} + 2\alpha_{\mu_i} < \min\{\frac{\delta}{2}, \frac{\gamma}{2}\} \leq \frac{\delta}{2}$, $P(u, v)$ is a $(\alpha_{\mu_p}, -d_N(\mu_p), d_S(\mu_p))$-path. In the latter case, assume, without loss of generality, that (u_p, v_p) has a negative projection on $d_N(\mu)$. Consider the path $P(u, v)$ from u to v that is a composition of the $(\frac{\alpha_\mu}{2}), -d_N(\mu))$-monotone path $P(u, u_p)$ between u and u_p, of the $(\alpha_{\mu_p}, -d_N(\mu_p), d_S(\mu_p))$-path $P(u_p, v_p)$ from u_p to v_p passing through w, which exists by induction, and of the $(\frac{\alpha_\mu}{2}), -d_N(\mu))$-monotone path $P(v_p, w')$ between v_p and a vertex w' adjacent to v in $skel(\mu)$, and of a $(\frac{\alpha_\mu}{2}), d_S(\mu))$-monotone path $P(w', v)$ from w' to v, which exists by induction. As $\beta_{\mu_i} + 2\alpha_{\mu_i} < \min\{\frac{\delta}{2}, \frac{\gamma}{2}\} \leq \frac{\delta}{2}$, $P(u, v)$ is a $(\alpha_{\mu_p}, -d_N(\mu_p), d_S(\mu_p))$-path. □

This concludes the proof of Theorem 4. □

6 Conclusions and Open Problems

In this paper we initiated the study of monotone graph drawings.

Concerning trees, we have shown that every monotone drawing is planar, that every strictly convex drawing is monotone, and that monotone drawings exist on polynomial-size grids. We believe that simple modifications of the algorithms we described allow one to construct strictly convex drawings of trees on polynomial-size grids. Another possible extension of our results is to characterize the monotonicity of a drawing in terms of the angles between adjacent edges. Our definition of slope-disjointness goes

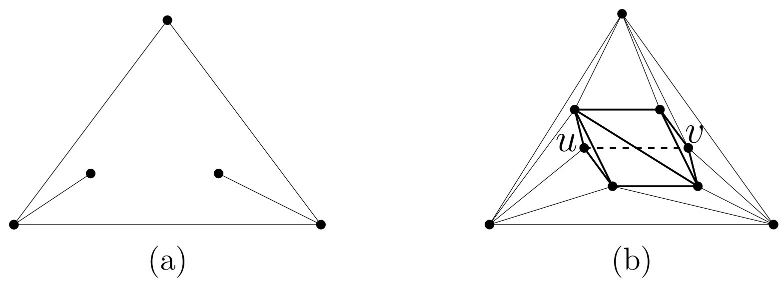

Figure 12: (a) A planar embedding of a graph with no monotone drawing. (b) A drawing of a planar triangulation that is not strongly monotone.

in this direction, although it introduces some non-necessary restrictions on the slopes of the edges (like the one that all the edge slopes are between 0 and π).

Further, we have proved that every biconnected planar graph admits a planar monotone drawing. Extending such a result to general simply-connected graphs seems to be non-trivial. Observe that there exist planar graphs that do not have a monotone drawing (see Fig. 12(a)) if the embedding is given. However, we are not aware of any planar graph not admitting a planar monotone drawing for any of its embeddings.

Several area minimization problems concerning monotone drawings are, in our opinion, worth of study. First, determining tight bounds for the area requirements of grid drawings of trees appears to be an interesting challenge. Second, modifying our tree drawing algorithms so that they construct grid drawings in general position would lead to algorithms for constructing monotone drawings of non-planar graphs on a grid of polynomial size. Third, the drawing algorithm we presented for biconnected planar graphs constructs drawings in which the ratio between the lengths of the longest and of the shortest edge is exponential in n. Is it possible to construct planar monotone drawings of biconnected planar graphs in polynomial area?

Finally, we introduce a new drawing standard which is definitely related to monotone drawings. A path from a vertex u to a vertex v is *strongly monotone* if it is monotone with respect to the half-line from u through v. A drawing of a graph G is *strongly monotone* if for each pair of vertices $u, v \in G$ a strongly monotone path $P(u, v)$ exists. Strong monotonicity appears to be even more desirable than general monotonicity for the readability of a drawing. However, designing algorithms for constructing strongly monotone drawings seems to be harder than for monotone drawings and it appears to be true that only restricted graph classes admit strongly monotone drawings. Note that a subpath of a strongly monotone path is, in general, not strongly monotone; notice also that while convexity implies monotonicity, it does not imply strong monotonicity, even for planar triangulations (see Fig. 12(b)).

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