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Upward Geometric Graph Embeddings into Point Sets

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ABSTRACT

We study the problem of characterizing the directed graphs with an upward straight-line embedding into every point set in general or in convex position. We solve two questions posed by Binucci *et al.* [*Computational Geometry: Theory and Applications, 2010*]. Namely, we prove that the classes of directed graphs with an upward straight-line embedding into every point set in convex position and with an upward straight-line embedding into every point set in general position do not coincide, and we prove that every directed caterpillar admits an upward straight-line embedding into every point set in convex position. Further, we provide new partial positive results on the problem of constructing upward straight-line embeddings of directed paths into point sets in general position.

1 Introduction

Constructing planar straight-line embeddings of graphs into point sets is a well-studied topic of research since more than twenty years. A celebrated result of Gritzmann *et al.* [9] is that the class of graphs that admit a planar straight-line embedding into every point set in general position or in convex position is the one of the *outerplanar graphs*. Efficient algorithms are known to embed outerplanar graphs [4] and trees [5] into any point set in general or in convex position. Further, while testing whether a graph admits a planar straight-line embedding into *every* point set in general or in convex position can be done efficiently, due to the above cited characterization [9] and to the existence of a linear-time algorithm to test whether a graph is outerplanar [11], testing whether a graph admits a planar straight-line embedding into a *given* point set in general position is $\mathcal{NP}$-hard, as proven by Cabello [6]. Planar graph embeddings into point sets have been also studied when edges are allowed to bend (see, e.g., [10, 2, 7]).

The problem of constructing upward planar straight-line embeddings of directed graphs into point sets has been first suggested by Giordano *et al.* [8] and has been very recently tackled by Binucci *et al.* in [3], who proved the following main results: (a) No biconnected directed graph admits an upward planar straight-line embedding into every point set in convex position; (ii) the upward planar straight-line embeddability of a directed graph into every *one-side convex point set* can be characterized and efficiently tested; (iii) there exist directed trees that do not have an upward planar straight-line embedding into every point set in convex position; (iv) every directed path admits an upward planar straight-line embedding into every point set in convex position.

In this paper we continue the study of the straight-line embeddability of directed graphs into planar point sets and show the following results.

In Sect. 3, we study upward planar straight-line embeddings of directed graphs into point sets in general and in convex position. First, we solve an open problem posed in [3], by exhibiting an infinite class of upward planar directed graphs that admit an upward planar straight-line embedding into every point set in convex position, but not into every point set in general position. Such a result shows an interesting difference between upward planar straight-line embeddability of directed graphs and planar straight-line embeddability of undirected graphs, since the classes of graphs with a planar straight-line embedding into every point set in convex position and with a planar straight-line embedding into every point set in general position coincide. Second, we show that every single-source upward planar directed graph with no cycle of length greater than three admits an upward planar straight-line embedding into every point set in general position. Such a result is the best possible with respect to the number of sources and to the length of the longest cycle.

In Section 4, we study upward planar straight-line embeddings of directed trees into point sets in convex position. We solve an open problem posed in [3] by proving that every directed caterpillar admits an upward planar straight-line embedding into every point set in convex position. This improves upon a previous result in [3] stating that every directed path admits an upward planar straight-line embedding into every point set in convex position.

In Section 5, we study upward planar straight-line embeddings of directed paths into point sets in general position. We tackle the problem by considering directed paths with few switches (a switch is either a source or a sink). While the upward planar straight-line embeddability of directed paths with at most two or three switches into point sets in general position can be trivially proven, it is already difficult to deal with directed paths with four or five switches. We prove that directed paths with four or five switches admit an upward planar straight-line embedding into every point set in general position, if we suppose that at least one and at least two of the monotone paths composing the directed paths with four or five switches, respectively, are single edges. Finally, we show that every directed path with at most k switches admits an upward planar straight-line embedding into every point set in general position with $n2^{k-2}$ points.

2 Preliminaries

A *point set in general position*, or *general point set*, is such that no three points lie on the same line and no two points have the same y-coordinate. The *convex hull* $Ch(S)$ of a point set S is the point set that can be obtained as a convex combination of the points of S. A *point set in convex position*, or *convex point set*, is such that no point is in the convex hull of the others. In a point set S, each point $p \in S$ is given by its coordinates $x(v)$ and $y(v)$ in the plane. We denote by $b(S)$ and by $t(S)$ the lowest and the highest point of S, respectively. A *one-side convex point set* S is a convex point set in which $b(S)$ and $t(S)$ are adjacent in the border of $Ch(S)$. During the execution of an algorithm which embeds a graph G into a point set S, a *free point* is a point of S to which no vertex of G has been mapped yet. Given a point p in a point set S, a subset S' of S is *clockwise separated around* p if a half-line fixed at p, starting from a horizontal position, directed towards decreasing x-coordinates, and moving clockwise encounters all the points of S' before encountering any other point of S. A *counterclockwise separation* is defined symmetrically.

An *upward planar directed graph* admits a planar drawing where each edge is represented by a curve monotonically increasing in the y-direction. In the following we refer to paths, cycles, caterpillars, and trees meaning upward planar directed graphs whose underlying graphs are paths, cycles, caterpillars, and trees, respectively.

An *upward straight-line embedding* of a graph into a point set is a mapping of each vertex to a distinct point and of each edge to a straight-line segment between its end-points such that no two edges cross and each edge (u, v) has $y(u) < y(v)$.

A *monotone path* $(v_1, v_2, \ldots, v_k)$ is such that edge (v_i, v_{i+1}) is directed from v_i to v_{i+1}, for $1 \leq i \leq k-1$. An upward straight-line embedding of a monotone path into any general point set S can be easily constructed by mapping vertex v_i to the i-th lowest point of S. A monotone path is *trivial* when it consists of a single edge.

3 Embeddings of Directed Graphs into Point Sets

In this section we study the relationship between upward straight-line graph embeddability into convex point sets and upward straight-line graph embeddability into general point sets, and the relationship between upward straight-line graph embeddability, the number of switches, and the length of the longest cycle in the underlying graph.

First, we show an infinite class of graphs that admit an upward straight-line embedding into every convex point set but not into every general point set.

Let G_k be defined as follows, for every $k \geq 3$: G_k has $3k$ vertices, it contains a 3-cycle C^3 composed of edges (u, v), (v, z), and (u, z), and it contains 4-cycles C_i^4, with $i = 1, \ldots, k-1$, composed of edges (u, v_i), (v_i, w_i), (w_i, z_i), and (u, z_i). See Fig. 1(a).

Lemma 1 *G_k admits an upward straight-line embedding into every convex point set S with $3k$ points.*

Proof: Map u to $b(S)$. Let l and r be the number of points in the subsets L and R of S to the left and to the right, respectively, of the line through $b(S)$ and $t(S)$.

If $l \equiv 0 \bmod 3$ (and $r \equiv 1 \bmod 3$), then iteratively map u_i, v_i, and z_i to the lowest three free points of L, for $i = 1, 2, \ldots, \frac{l}{3}$; further, iteratively map u_i, v_i, and z_i to the lowest three free points of R, for $i = \frac{l+3}{3}, \frac{l+6}{3}, \ldots, k-1$; finally, map v and z to the highest point of R and to $t(S)$, respectively. If $l \equiv 1 \bmod 3$ (and $r \equiv 0 \bmod 3$), then iteratively map u_i, v_i, and z_i to the lowest three free points of L, for $i = 1, 2, \ldots, \frac{l-1}{3}$; further, map v and z to the highest point of L and to $t(S)$, respectively; finally, iteratively map u_i, v_i, and z_i to the lowest three free points of R, for $i = \frac{l+2}{3}, \frac{l+5}{3}, \ldots, k-1$. If $l \equiv 2 \bmod 3$ (and $r \equiv 2 \bmod 3$), then iteratively map u_i, v_i, and z_i to the lowest three free points of $L \cup \{t(S)\}$, for $i = 1, 2, \ldots, \frac{l+1}{3}$; further, iteratively map u_i, v_i, and z_i to the lowest three free points of R, for $i = \frac{l+4}{3}, \frac{l+7}{3}, \ldots, k-1$; finally, map v and z to the highest two points of R.

In all the cases, the obtained straight-line embedding is upward and planar. □

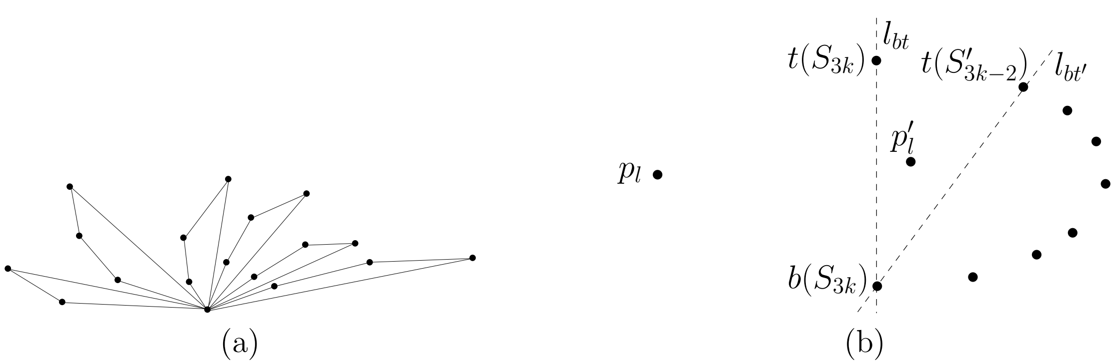

Figure 1: (a) Graph G_k. (b) Point set S_{3k}.

Lemma 2 *There exists a general point set S_{3k} with $3k$ points such that G_k does not admit any upward straight-line embedding into S_{3k}.*

Proof: Point set S_{3k} is any point set that satisfies the following constraints (see Fig. 1(b)). One point p_l is to the left of the line l_{bt} through $b(S_{3k})$ and $t(S_{3k})$. The remaining $3k-3$ points are to the right of l_{bt} and, together with $b(S)$, they form a convex point set S'_{3k-2} with one point p'_l lying to the left of the line $l_{bt'}$ through $b(S)$ and $t(S'_{3k-2})$.

Observation 1 *Let G be a graph containing a 4-cycle C composed of edges (x_1, x_2), (x_2, x_3), (x_3, x_4), and (x_1, x_4). Let S be a point set such that exactly one point $p_l(S)$ lies to the left of the line through $b(S)$ and $t(S)$. Suppose that an edge of G has been mapped to segment $\overline{b(S)t(S)}$. Then, there exists no upward embedding of G into S in which a vertex of C is mapped to $p_l(S)$.*

Proof: Suppose that a vertex of C is mapped to $p_l(S)$. If such a vertex is x_4, then x_3 is mapped to a point to the right of $l_{bt}(S)$ (as a source of G has to be mapped to $b(S)$), hence $\overline{b(S)t(S)}$ crosses the segment between x_4 and x_3; if such a vertex is x_3, then x_2 is mapped to a point to the right of $l_{bt}(S)$ (as a source of G has to be mapped to $b(S)$), hence $\overline{b(S)t(S)}$ crosses the segment between x_3 and x_2; if such a vertex is x_2, then vertex x_3 is mapped to a point to the right of $l_{bt}(P)$ (as a sink of G has to be mapped to $t(S)$), hence $\overline{b(S)t(S)}$ crosses the segment between x_2 and x_3; if such a vertex is x_1, then vertex x_2 is mapped to a point to the right of $l_{bt}(S)$ (as a sink of G has to be mapped to $t(S)$), hence $\overline{b(S)t(S)}$ crosses the segment between x_1 and x_2. The proof of the observation follows. □

Since u is the only source of G_k, such a vertex has to be mapped to $b(S_{3k})$. Further, since a sink of G_k has to be mapped to $t(S_{3k})$ and since every sink of G_k is adjacent to u, segment $\overline{b(S_{3k})t(S_{3k})}$ is part of any embedding. Then, by Observation 1 no vertex of a 4-cycle C^4_i of G_k is mapped to p_l. Hence, a vertex of C^3 is mapped to p_l. If such a vertex is z, then vertex v is mapped to a point to the right of l_{bt}, hence $\overline{b(S)t(S)}$ crosses the segment between z and v. It follows that v is mapped to p_l. If z is not mapped to $t(S_{3k})$, then $\overline{b(S_{3k})t(S_{3k})}$ crosses the segment between z and v. Hence, z is mapped to $t(S_{3k})$. Then, all the vertices of the 4-cycles of G_k are mapped to the vertices of S'_{3k-2}. A sink of one of the 4-cycles has to be mapped to $t(S'_{3k})$. Since every sink of G_k is adjacent to u, segment $\overline{b(S_{3k})t(S'_{3k})}$ is part of any embedding. Then, by Observation 1 no vertex of a 4-cycle C^4_i of G_k can be mapped to p'_l, thus proving the lemma. □

We get the following:

Theorem 1 *For every $k \geq 3$, there exists a $3k$-vertex upward planar digraph that admits an upward straight-line embedding into every convex point set with $3k$ points but not into every general point set with $3k$ points.*

Next, we show that every single-source graph G whose every simple cycle has length three admits an upward straight-line embedding into every general point set. Such a result is tight both with respect to the maximum length of a cycle in G and with respect to the number of sources in G. Namely, by Lemma 2, a single-source graph exists whose every simple cycle has length at most four not admitting any upward straight-line embedding into some general point set. Further, by the results in [3] on upward straight-line embeddability into one-side convex point sets, there exists a graph with two sources whose every simple cycle has length three not admitting any upward straight-line embedding into some general point set (see Fig. 2).

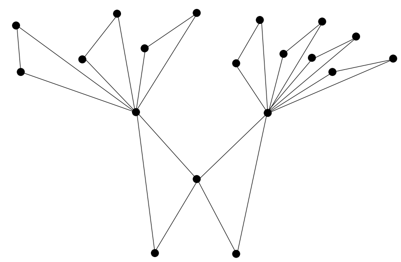

Figure 2: A graph with two sources whose every simple cycle has length three.

We show a recursive algorithm to construct upward straight-line embeddings of single-source graphs whose every simple cycle has length three into every general point set. The recursion is on the number x of biconnected components of G. If $x = 1$, then the statement is trivially true. If $x > 1$, consider the unique source s of G.

If s is a cutvertex of G, denote by $B_1, \dots, B_k$ the connected components obtained by removing s from G (see Fig. 3(a)). Clockwise separate sets $S_1, \dots, S_k$ with $|B_1|, \dots, |B_k|$ points, respectively, around $b(S)$. Recursively construct straight-line embeddings of the subgraphs of G induced by the vertices in $B_1 \cup \{b(S)\}$, …, in $B_k \cup \{b(S)\}$ into point sets $S_1 \cup \{b(S)\}$, …, $S_k \cup \{b(S)\}$, respectively.

If s is not a cutvertex of G, consider the biconnected component B incident to s.

If B is an edge (s, s'), denote by S' the point set obtained by removing $b(S)$ from S and by G' the graph obtained by removing s and its incident edge from G (see Fig. 3(b)). Recursively construct an upward straight-line embedding of G' into S'.

If B is a 3-cycle (s, s', s''), denote by $b^*(S)$ the lowest point of S different from $b(S)$. Denote by G' (by G'') the graph composed of s' (resp. of s'') and of every connected component not containing s that is obtained by removing s' (resp. s'') from G. Clockwise separate $|G'| - 1$ points around $b^*(S)$. Such points, together with $b^*(S)$, form a set S'. Let $S'' = S \setminus \{S' \cup \{b(S)\}\}$. Consider a line l fixed at $b^*(S)$ and rotating clockwise starting from a horizontal position. If l encounters $b(S'')$ after s (see Fig. 3(c)), then recursively construct upward straight-line embeddings of G' into S' and of G'' into S''. If l encounters $b(S'')$ before s (see Fig. 3(d)), then counterclockwise separate $|G'| - 1$ points around $b^*(S)$. Such points, together with $b^*(S)$, form a set S_1. Let $S_2 = S \setminus \{S_1 \cup \{b(S)\}\}$. Then, recursively construct upward straight-line embeddings of G' into S_1 and of G'' into S_2.

We get the following:

Theorem 2 *Every single-source upward planar directed graph whose every simple cycle has length three admits an upward straight-line embedding into every point set in general position.*

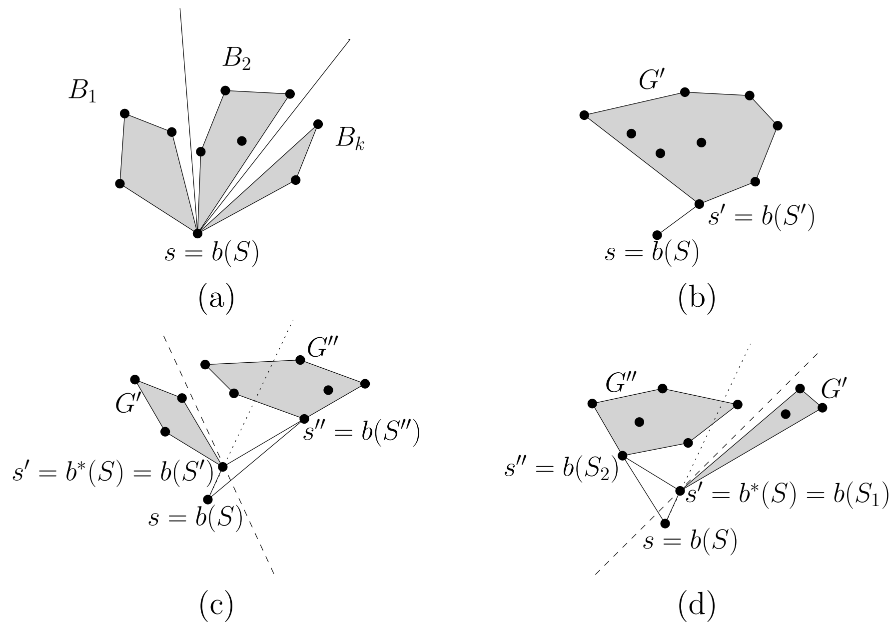

Figure 3: (a) s is a cut-vertex. (b) B is an edge. (c) B is a 3-cycle (s, s', s'') and l encounters $b(S'')$ after s. (d) B is a 3-cycle (s, s', s'') and l encounters $b(S'')$ before s.

Proof: We prove that the recursive algorithm described in Section 3 constructs an upward straight-line embedding of G into S.

If s is a cutvertex of G, then the constructed drawing Γ of G is upward and straight-line as the drawings of the subgraphs $B_i \cup \{s\}$ of G are. Further, the drawings of such subgraphs lie into disjoint convex regions of the plane, hence Γ is planar.

If s is not a cutvertex of G and the unique biconnected component B incident to s is an edge (s, s'), then the constructed drawing Γ of G is straight-line by construction, is upward since the graph G' obtained by removing from G vertex s and edge (s, s') is drawn upward by recursion and since edge (s, s') is drawn upward as $b(S)$ is the bottommost point of S, and is planar since G' is drawn planar by recursion and since drawing edge (s, s') as a segment does not cause crossings, as such a segment lies completely below the smallest horizontal strip containing S', except for its extremal point $b(S')$.

If s is not a cutvertex of G, if the unique biconnected component B incident to s is a 3-cycle (s, s', s''), and if the sweeping line l encounters $b(S'')$ after s when rotating clockwise starting from a horizontal position, then the constructed drawing Γ of G:

- is straight-line, by construction;
- is upward, since G' and G'' are drawn upward by recursion (observe that $b(S')$ and $b(S'')$ are the bottommost points of S' and S'', respectively) and since edges (s, s'), (s, s''), and (s', s'') are upward as $y(b(S)) < y(b(S')) < y(b(S''))$; and
- is planar, since the drawings of G' and G'' do not cross as they are separated

by l, since drawing edge (s, s') as a segment does not cause crossings, as such a segment lies completely below the smallest horizontal strip containing S' and S'', except for its extremal point $b^*(S) = b(S')$, since drawing edge (s', s'') as a segment does not cause crossings, as such a segment lies completely below the smallest horizontal strip containing S'', except for its extremal point $b(S'')$, and since drawing edge (s, s'') as a segment does not cause crossings, as such a segment lies completely below the smallest horizontal strip containing S'', except for its extremal point $b(S'')$, and as l encounters $b(S'')$ after s when rotating clockwise.

If s is not a cutvertex of G, if the unique biconnected component B incident to s is a 3-cycle (s, s', s''), and if the sweeping line l encounters $b(S'')$ before s when rotating clockwise starting from a horizontal position, line l encounters $b(S_2)$ after s when rotating clockwise. Namely, if $b(S'') \in S_1$, then all the points of S_2 (hence also $b(S_2)$) are encountered by l after s. Otherwise, $b(S'') \in S_2$ and all the points of S_2 that are encountered by l before s, if any, also belong to S'', as they are encountered by l after $b(S'')$. Hence, such points lie above $b(S'')$, which implies that none of them can be $b(S_2)$. The constructed drawing Γ of G:

- is straight-line, by construction;
- is upward, since G' and G'' are drawn upward by recursion (observe that $b(S_1)$ and $b(S_2)$ are the bottommost points of S_1 and S_2, respectively) and since edges (s, s'), (s, s''), and (s', s'') are upward as $y(b(S)) < y(b(S_1)) < y(b(S_2))$; and
- is planar, since the drawings of G' and G'' do not cross as they are separated by l, since drawing edge (s, s') as a segment does not cause crossings, as such a segment lies completely below the smallest horizontal strip containing S_1 and S_2, except for its extremal point $b^*(S) = b(S_1)$, since drawing edge (s', s'') as a segment does not cause crossings, as such a segment lies completely below the smallest horizontal strip containing S_2, except for its extremal point $b(S_2)$, and since drawing edge (s, s'') as a segment does not cause crossings, as such a segment lies completely below the smallest horizontal strip containing S_2, except for its extremal point $b(S_2)$, and as l encounters $b(S'')$ before s when rotating clockwise.

□

4 Embedding Directed Caterpillars into Convex Point Sets

In this section we prove that every caterpillar admits an upward straight-line embedding into every point set in convex position.

We introduce some terminology. A *caterpillar* G is a tree such that removing all the degree-1 vertices, called the *legs* of G, yields a path, called the *spine* of G. A caterpillar whose spine is a monotone path is a *monotone caterpillar*. Let v_s and v_t be a source and a sink of the spine of a caterpillar. A vertex w that is connected to v_s by edge (w, v_s) or to v_t by edge (v_t, w) is an *extremal leg* of a caterpillar. In Fig. 4(a) the extremal legs are numbered $1, 13, 14, 20, 21, 28, 29, 39, 40, 41$.

Let G be a caterpillar, let T be its spine, and let u be one of the end-points of T. Let U be the set composed of u and of the extremal legs of G adjacent to u. The following lemma descends from algorithms presented in the literature [8, 3].

Lemma 3 *Suppose that u is a source (resp. a sink) of T. Then, G admits an upward straight-line embedding into every one-side convex point set S in which the vertices of G in U are mapped to the $|U|$ lowest (resp. highest) points of S.*

Let G be a monotone caterpillar and let T be its spine. Let s and t be the source and the sink of T, respectively. In addition, suppose that s and t are a source and a sink of G, respectively. We have the following:

Lemma 4 *G admits an upward straight-line embedding into every convex point set S in which s is mapped to the lowest point of S and t is mapped to the highest point of S.*

Proof: Let $v_1, \dots, v_n$ be a vertex ordering of G such that $\mathcal{E}(v_i) = i$. Suppose that $v_1 = s$ and $v_n = t$, the case in which $v_1 = t$ and $v_n = s$ being analogous. Map vertex v_i to the i-th lowest point of S. Then s and t are mapped to the lowest and highest point of S, respectively. Notice that $v_1, \dots, v_n$ is a topological ordering of G, hence the embedding is upward. Finally, the embedding is planar since, for any two vertices u and v of T, the smallest horizontal strip containing u and its adjacent legs is disjoint from the smallest horizontal strip containing v and its adjacent legs. □

Next, for a caterpillar $G = (V, E)$, we define a bijective function $\mathcal{E} : V \to \{1, 2, \dots, n\}$. Let T be the spine of G and let a and b be the end-vertices of T. Function $\mathcal{E}$ is defined according to the following rules (see Fig. 4). ($\mathcal{R}1$): For any two vertices $u, v \in T$ such that u comes before v when traversing T from a to b, the value associated to u and to all the legs adjacent to u is smaller than the value associated to v and to all legs adjacent to v; ($\mathcal{R}2$): For any vertex $u \in T$, the value associated to u is greater than the value associated to all the legs incident to edges entering u; ($\mathcal{R}3$): For any vertex $u \in T$, the value associated to u is smaller than the value associated to all the legs incident to edges exiting u.

We now describe an algorithm to construct an upward straight-line embedding of any caterpillar G into any convex point set S. The idea is to partition G into three smaller caterpillars that can be embedded into suitable subsets of S by means of the two algorithms described above. In the following we formalize this idea.

Let G be a caterpillar with spine T. Let $(T_1, \dots, T_k)$ be the maximal monotone paths composing T; T_i is an *increasing path* if, for any edge (u, v) in T_i, $\mathcal{E}(u) < \mathcal{E}(v)$ and a *decreasing path* otherwise. Assume the sources and the sinks of T belong to

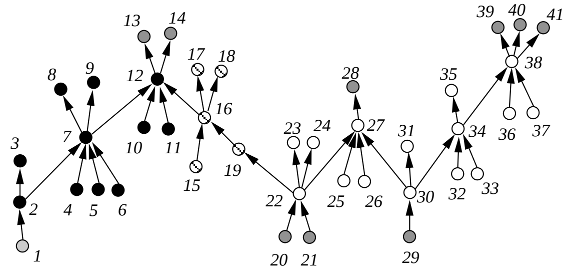

Figure 4: A caterpillar $G = (V, E)$ and function $\mathcal{E} : V \to \{1, 2, \ldots, n\}$.

the increasing paths and not to the decreasing paths. Hence, T_1 and T_k are increasing paths, possibly with one vertex. Let R_i be the caterpillar induced by T_i and by the non-extremal legs of G adjacent to T_i; R_i is an *increasing* (resp. *decreasing*) *caterpillar* if T_i is an increasing (resp. decreasing) path. Caterpillar G is partitioned into increasing caterpillars, decreasing caterpillars, and extremal legs (see Fig. 5). If R_i is an increasing caterpillar, let $s(R_i)$ and $t(R_i)$ be the source and the sink of T_i, respectively. If R_i is a decreasing caterpillar, let $t(R_i) = t(R_{i-1})$ and $s(R_i) = s(R_{i+1})$, that is, $s(R_i)$ (resp. $t(R_i)$) is the source (the sink) of T immediately following (preceding) T_i. Observe that, if R_i is a decreasing caterpillar, $s(R_i)$ and $t(R_i)$ do not belong to R_i.

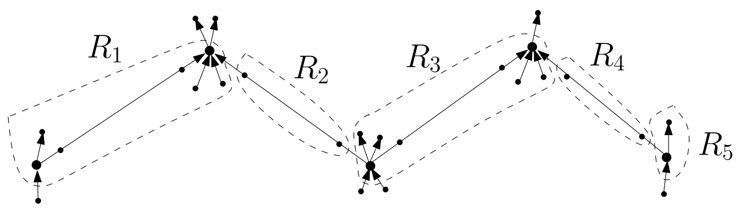

Figure 5: Decomposition of a caterpillar into increasing caterpillars, decreasing caterpillars, and extremal legs.

Next, we define the *sub-caterpillars* G_i^1 *and* G_i^2 *of* G *induced by* R_i. If R_i is an increasing caterpillar (see Fig. 6(a)), set G_i^1 (G_i^2) to be the caterpillar induced by the vertices of G preceding $s(R_i)$ (following $t(R_i)$, resp.) in $\mathcal{E}$, except for the extremal legs adjacent to $s(R_i)$ (to $t(R_i)$, resp.). If R_i is a decreasing caterpillar (see Fig. 6(b)), set G_i^1 (G_i^2) to be the caterpillar induced by $t(R_i)$ (by $s(R_i)$, resp.) and by the vertices of G preceding $t(R_i)$ (following $s(R_i)$, resp.) in $\mathcal{E}$.

Consider a line l through $b(S)$ and $t(S)$. Denote by A and B the point sets to the left and to the right of l, resp. Points $b(S)$ and $t(S)$ belong to A. Consider any increasing caterpillar R_i. Denote by i_l (resp. i_h) the number of extremal legs adjacent to $s(R_i)$ (resp. to $t(R_i)$) and by L (resp. H) the set of the $i_l + 1$ lowest (resp. of the $i_h + 1$ highest) points of S. Let $A' = A \setminus (L \cup H)$, $B' = B \setminus (L \cup H)$, $|H \cap A| = h_a$, $|H \cap B| = h_b$, $|L \cap A| = l_a$, and $|L \cap B| = l_b$ (see Fig. 7).

We have the following:

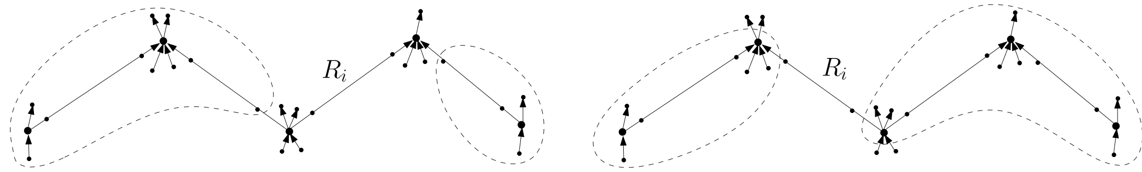

Figure 6: Sub-caterpillars of (a) an increasing caterpillar R_i and (b) a decreasing caterpillar R_i.

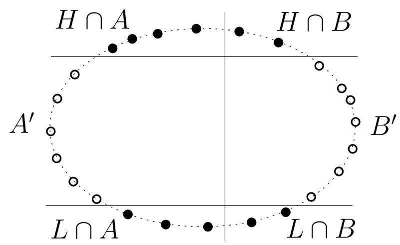

Figure 7: Partition of S.

Lemma 5 *If $|G_i^1| \leq |A| - l_a$ and $|G_i^2| \leq |B| - h_b$, then there is an upward straight-line embedding of G into S.*

Proof: We distinguish three cases:

Case 1*:* $|G_i^1| \leq |A'|$ and $|G_i^2| \leq |B'|$. Refer to Fig. 8. Map $s(R_i)$ to $t(L)$ and the extremal legs adjacent to $s(R_i)$ to the other points of L. Map $t(R_i)$ to $b(H)$ and the extremal legs adjacent to $t(R_i)$ to the other points of H. Embed G_i^1 into the $|G_i^1|$ lowest points of A' and G_i^2 into the $|G_i^2|$ highest points of B'. Such embeddings can be constructed by Lemma 3, since A' and B' are one-side convex point sets. Embed R_i into the remaining free points of S. This can be done by Lemma 4, since R_i is a monotone caterpillar.

We have the following:

Claim 1 *The constructed straight-line embedding of G into S is upward and planar.*

Proof: We prove the upwardness. Since $s(R_i)$ is mapped to the highest point of L and since all the extremal legs adjacent to $s(R_i)$ are mapped to points of L, the edges

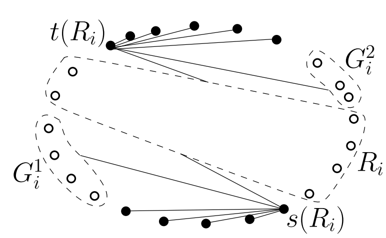

Figure 8: Embedding G into S if (a) $|G_i^1| \leq |A'|$ and $|G_i^2| \leq |B'|$.

connecting such legs to $s(R_i)$ are upward. Analogously, the edges connecting $t(R_i)$ to the extremal legs adjacent to $t(R_i)$ are upward. The embeddings of G_i^1 and G_i^2 are upward by Lemma 3 and the embedding of R_i is upward by Lemma 4.

We prove the planarity. The edges connecting $s(R_i)$ to the extremal legs adjacent to $s(R_i)$ and the edges connecting $t(R_i)$ to the extremal legs adjacent to $t(R_i)$ lie in the smallest horizontal strips containing L and H, respectively. Such edges do not cross any other edge of G, since no other edge of G has intersection with the interior of such strips. The embeddings of G_i^1 and G_i^2 are planar by Lemma 3, and the embedding of R_i is planar by Lemma 4. Further, such embeddings do not cross as they lie into disjoint convex regions of the plane. □

Case 2: $|G_i^1| > |A'|$. We create a partition (A'_1, B'_1, H_1, L_1) of S such that $|G_i^1| = |A'_1|$, $|G_i^2| + |R_i| = |B'_1|$, $i_l + 1 = |L_1|$, and $i_h + 1 = |H_1|$ (see Fig. 9(a)). Let $d_{G_i^1} = |G_i^1| - |A'|$. By the assumptions of the lemma, $|G_i^1| \leq |A| - l_a$. Since $|A| = |A'| + l_a + h_a$, we have $d_{G_i^1} \leq h_a$. Define A'_1 as A' plus the $d_{G_i^1}$ lowest points of $H \cap A$, B'_1 as B' minus the $d_{G_i^1}$ highest points of B', $L_1 = L$, and $H_1 = S \setminus (A'_1 \cup B'_1 \cup L_1)$. Refer to Fig. 9(b). Map $s(R_i)$ to $t(L_1)$ and map the extremal legs adjacent to $s(R_i)$ to the other points of L_1. Map $t(R_i)$ to $b(H_1)$ and the extremal legs adjacent to $t(R_i)$ to the other points of H_1. Embed G_i^1 into A'_1 and G_i^2 into the $|G_i^2|$ highest points of B'_1. Such embeddings can be constructed by Lemma 3, since A'_1 and B'_1 are one-side convex point sets. Embed R_i into the remaining free points of S. This can be done by Lemma 4, since R_i is a monotone caterpillar.

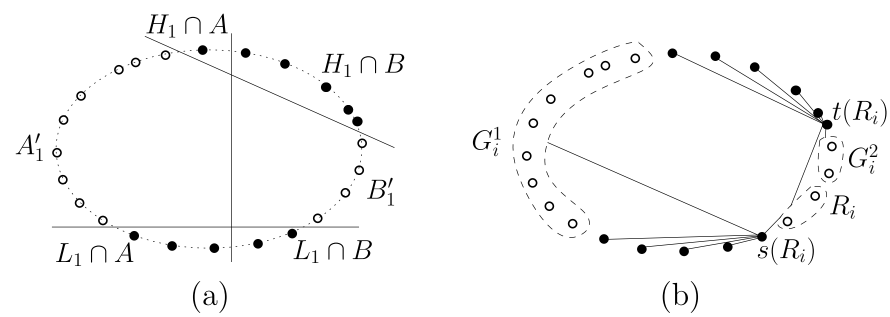

Figure 9: (a) Point set for Case 2. (b) Embedding G into S if $|G_i^1| \leq |A| - l_a$, $|G_i^2| \leq |B| - h_b$, and $|G_i^1| > |A'|$.

We have the following:

Claim 2 *The constructed straight-line embedding of G into S is upward and planar.*

Proof: We prove the upwardness. Since $s(R_i)$ is mapped to the highest point of L and since all the extremal legs adjacent to $s(R_i)$ are mapped to points of L, the edges connecting such legs to $s(R_i)$ are upward. Analogously, the edges connecting $t(R_i)$ to the extremal legs adjacent to $t(R_i)$ are upward. The embeddings of G_i^1 and G_i^2 are upward by Lemma 3 and the embedding of R_i is upward by Lemma 4.

We prove the planarity. The edges connecting $s(R_i)$ to the extremal legs adjacent to $s(R_i)$ lie in the smallest horizontal strip containing L_1. Such edges do not cross any other edge of G, since no other edge of G has intersection with the interior of such a strip. The edges connecting $t(R_i)$ to the extremal legs adjacent to $t(R_i)$ lie in the half-plane delimited by a line separating H_1 from the other points of S. Such edges do not cross any other edge of G, since no other edge of G has intersection with the interior of such a half-plane. The embeddings of G_i^1 and G_i^2 are planar by Lemma 3, and the embedding of R_i is planar by Lemma 4. Further, such embeddings do not cross as they lie into disjoint convex regions of the plane. □

Case 3*:* $|G_i^1| \leq |A'|$ and $|G_i^2| > |B'|$. We create a partition (A'_1, B'_1, H_1, L_1) of S such that $|G_i^1| + |R_i| = |A'_1|$, $|G_i^2| = |B'_1|$, $i_l + 1 = |L_1|$, and $i_h + 1 = |H_1|$ (see Fig. 10(a)). Let $d_{G_i^2} = |G_i^2| - |B'|$. By the assumptions of the lemma, $|G_i^2| \leq |B| - h_b$. Since $|B| = |B'| + l_b + h_b$, we have $d_{G_i^2} \leq l_b$. Define B'_1 as B' plus the $d_{G_i^2}$ highest points of $L \cap B$, A'_1 as A' minus the $d_{G_i^2}$ lowest points of A', $H_1 = H$, and $L_1 = S \setminus (A'_1 \cup B'_1 \cup H_1)$. Refer to Fig. 10(b). Map $s(R_i)$ to $t(L_1)$ and the extremal legs adjacent to $s(R_i)$ to the other points of L_1. Map $t(R_i)$ to $b(H_1)$ and the extremal legs adjacent to $t(R_i)$ to the other points of H_1. Embed G_i^2 into B'_1 and G_i^1 into the $|G_i^1|$ lowest points of A'_1. Such embeddings can be constructed by Lemma 3, since A'_1 and B'_1 are one-side convex point sets. Embed R_i into the remaining free points of S. This can be done by Lemma 4, since R_i is a monotone caterpillar.

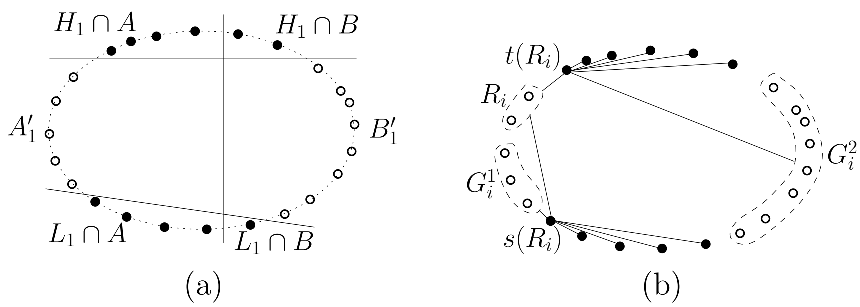

Figure 10: (a) Point set for Case 3. (b) Embedding G into S if $|G_i^2| \leq |B| - h_b$, $|G_i^1| \leq |A'|$, and $|G_i^2| > |B'|$.

We have the following:

Claim 3 *The constructed straight-line embedding of G into S is upward and planar.*

Proof: The proof is analogous to the proof of Claim 2. □

□

Now consider any decreasing caterpillar R_i that is part of G. Define i_l, i_h, L, H, A', B', h_a, h_b, l_a, and l_b as before. We have the following:

Lemma 6 *If $|G_i^1| \leq |A| - h_a + 1$ and $|G_i^2| \leq |B| - l_b + 1$, then there is an upward straight-line embedding of G into S.*

The proof of Lemma 6 is analogous to the one of Lemma 5. The requirements $|G_i^1| \leq |A| - h_a + 1$ and $|G_i^2| \leq |B| - l_b + 1$ of Lemma 6 are weaker than the requirements $|G_i^1| \leq |A| - l_a$ and $|G_i^2| \leq |B| - h_b$ of Lemma 5. This is due to the fact that, if R_i is a decreasing caterpillar, then $t(R_i)$ and $s(R_i)$ belong to G_i^1 and to G_i^2, respectively, and they do not belong to R_i. We are now ready to prove the following:

Theorem 3 *Any n-vertex directed caterpillar G admits an upward straight-line embedding into every convex point set S with n points.*

Proof: Let T be the spine of G and let $\{R_1, \dots, R_k\}$ be the increasing and decreasing caterpillars of G. Let (A, B) be the partition of S created by a line through $b(S)$ and $t(S)$, where $b(S)$ and $t(S)$ belong to A. Consider the $|A|$-th vertex $v_{|A|}$ in the order $v_1, \dots, v_n$ of the vertices of G defined by $\mathcal{E}$. We partition G into three smaller caterpillars and we draw each of them on a suitably chosen portion of S. The partition of G is determined by the position of $v_{|A|}$ in G. We distinguish four cases:

Case 1: $v_{|A|}$ is a vertex of an increasing caterpillar R_i. Define i_l, i_h, L, H, h_b, l_a, h_a, and l_b as before. Since $v_{|A|} \in R_i$, we have that $|G_i^1| + i_l < |A|$. Since $i_l + 1 = l_a + l_b$, it follows that $l_a \leq i_l + 1$. Thus, $|G_i^1| + l_a \leq |A|$. Analogously, we have that $|G_i^2| + i_h < |B|$. Since $i_h + 1 = h_a + h_b$, it follows that $h_b \leq i_h + 1$. Thus, $|G_i^2| + h_b \leq |B|$. Hence, Lemma 5 applies and the result follows.

Case 2: $v_{|A|}$ is a vertex of a decreasing caterpillar R_i. Analogously to Case 1, it can be proven that $|G_i^1| \leq |A| - h_a + 1$ and $|G_i^2| \leq |B| - l_b + 1$. Hence, Lemma 6 applies and the result follows.

Case 3: $v_{|A|}$ is an extremal leg adjacent to a sink of T. Let R_i and R_{i+1} be such that $t(R_i) = t(R_{i+1})$ and $v_{|A|}$ is an extremal leg adjacent to $t(R_i)$. Note that R_i is an increasing caterpillar and R_{i+1} is a decreasing caterpillar. Denote by i_h the number of extremal legs adjacent to $t(R_i)$ and by H the set of the $i_h + 1$ highest points of S. Let $|H \cap B| = h_b$ and $|H \cap A| = h_a$. Notice that $h_a + h_b = i_h + 1$. We claim the following.

Claim 4 *Let G_i^1 and G_i^2 (let G_{i+1}^1 and G_{i+1}^2) be the sub-caterpillars of G induced by R_i (resp. by R_{i+1}). At least one of the following inequalities holds: (1) $|G_{i+1}^1| \leq |A| - h_a + 1$; (2) $|G_i^2| \leq |B| - h_b$.*

Proof: Suppose, for a contradiction, that both inequalities do not hold. Then $|A| + |B| < |G_{i+1}^1| + |G_i^2| + h_a + h_b - 1 = |G_{i+1}^1| + |G_i^2| + i_h = |G|$. Hence $|A| + |B| < |G|$, a contradiction. □

We further distinguish two cases.

Inequality (1) holds. Consider the decreasing caterpillar R_{i+1}. Let i_l be the number of extremal legs adjacent to $s(R_{i+1})$ and let L be the set of the $i_l + 1$ lowest point of S. Denote $l_a = |L \cap A|$ and $l_b = |L \cap B|$. We have that $|B| > |G_{i+1}^1| + i_l$, hence $|B| \geq |G_{i+1}^1| + l_b$, as $i_l + 1 = l_a + l_b$. Hence, Lemma 6 applies and the result follows.

Inequality (2) holds. Consider the increasing caterpillar R_i. Let i_l be the number of extremal legs adjacent to $s(R_{i+1})$ and let L be the set of the $i_l + 1$ lowest point of S. Denote $l_a = |L \cap A|$ and $l_b = |L \cap B|$. We have that $|A| > |G_i^1| + i_l$, hence $|A| \geq |G_i^1| + l_a$, as $i_l + 1 = l_a + l_b$. Hence, Lemma 5 applies and the result follows.

Case 4: $v_{|A|}$ is an extremal leg adjacent to a source of T. Analogously to Case 3, it can be proven that either $|A| \geq |G_{i+1}^1| + h_a - 1$ and $|B| \geq |G_{i+1}^1| + l_b - 1$ hold simultaneously, thus the result follows from Lemma 6, or $|A| \geq |G_i^1| + l_a$ and $|B| \geq |G_i^2| + h_b$ hold simultaneously, thus the result follows from Lemma 5. □

5 Embedding Directed Paths into General Point Sets

In this section we deal with upward straight-line embeddings of paths with few switches into general point sets. We first deal with paths with four switches.

Theorem 4 *Every path P composed of three monotone paths P_1, P_2, and P_3 admits an upward straight-line embedding into every general point set S if at least one out of P_1, P_2, and P_3 is a trivial path.*

Proof: Let $P_1 = (s_1 = u_1, \ldots, u_U = t_1)$, $P_2 = (t_1 = v_1, \ldots, v_V = s_2)$, and $P_3 = (s_2 = w_1, \ldots, w_W = t_2)$ be the monotone paths composing P, where s_1 and s_2 are sources and t_1 and t_2 are sinks.

If P_2 is trivial, a more general result in [3] states that a path P admits an upward straight-line embedding into every general point set if the i-th monotone path composing P is trivial, for every odd i or for every even i.

We discuss the case in which P_3 is trivial, the case in which P_1 is trivial being symmetric. Counterclockwise separate a set S_1 of $U-1$ points around $t(S)$. Construct upward straight-line embeddings of P_1 into $S_1 \cup \{t(S)\}$ and of $P_2 \setminus \{s_2\}$ into the $V-1$ highest points of $S \setminus S_1$. Map s_2 to $b(S \setminus S_1)$ and t_2 to the only remaining free point of S.

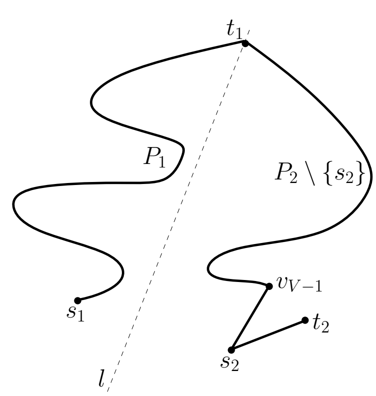

Figure 11: Embedding a path with four switches whose third monotone path is trivial into a general point set.

Claim 5 *The constructed straight-line embedding of P into S is upward and planar.*

Proof: Refer to Fig. 11. The constructed straight-line embedding is upward as P_1, P_2, and P_3 are drawn upward by construction. Further, it is planar, namely the drawing of P_1 lies entirely to the left of l, while the drawing of P_2 lies entirely to the right of l; further, the only edge of P_2 that crosses the smallest horizontal strip containing (s_2, t_2) is (v_{V-1}, v_V). However, such an edge is adjacent to (s_2, t_2), hence they do not cross. □

□

We now deal with paths with five switches.

Theorem 5 *Every path P composed of four monotone paths P_1, P_2, P_3, and P_4 admits an upward straight-line embedding into every general point set S if at least two out of P_1, P_2, P_3, and P_4 are trivial paths.*

Proof: Let $P_1 = (s_1 = u_1, u_2, \ldots, u_U = t_1)$, $P_2 = (t_1 = v_1, v_2, \ldots, v_V = s_2)$, $P_3 = (s_2 = w_1, w_2, \ldots, w_W = t_2)$, and $P_4 = (t_2 = z_1, z_2, \ldots, z_Z = s_3)$ be the monotone paths composing P, where s_1, s_2, and s_3 are sources and t_1 and t_2 are sinks. The case in which P has three sinks and two sources can be discussed analogously.

If P_1 and P_3 are trivial or if P_2 and P_4 are trivial, the proof follows from the result in [3] cited in the proof of Theorem 4.

We discuss the case in which P_1 and P_4 are trivial. Clockwise separate a set $S_{1,2}$ of V points around $b(S)$. Construct upward straight-line embeddings of $P_2 \setminus \{t_1\}$ into the $V-1$ lowest points of $S_{1,2} \cup b(S)$ and of $P_3 \setminus \{t_2\}$ into the $W-1$ lowest points of $S \setminus S_{1,2}$. Map t_1 to $t(S_{1,2})$ and s_1 to the only remaining free point of $S_{1,2}$. Map t_2 to $t(S \setminus S_{1,2})$ and s_3 to the only remaining free point of $S \setminus S_{1,2}$.

Claim 6 *The constructed straight-line embedding of P into S is upward and planar.*

Proof: Refer to Fig. 12(a). The constructed straight-line embedding is upward as P_1, P_2, P_3, and P_4 are drawn upward by construction. Further, it is planar, namely the drawings of P_1 and P_2 lie entirely to the left of l, while the drawings of P_3 and P_4 lie entirely to the right of l; further, the only edge of P_2 that crosses the smallest horizontal strip containing (s_1, t_1) is (v_1, v_2). However, such an edge is adjacent to (s_1, t_1), hence they do not cross. Analogously, P_3 and P_4 do not cross. □

We discuss the case in which P_1 and P_2 are trivial, the case in which P_3 and P_4 are trivial being symmetric. Clockwise separate a set S_4 of $Z-1$ points around $t(S)$. Construct upward straight-line embeddings of P_4 into $S_4 \cup \{t(S)\}$ and of $P_3 \setminus \{s_2\}$ into the $W-1$ highest points of $S \setminus S_4$. Consider the line l' through the point where w_2 is drawn and the lowest free point of S. If both the two remaining free points of S are on the same side of l', then map s_2 to $b(S \setminus S_4)$, map t_1 to the highest free point of S, and map s_1 to the other free point of S. Otherwise, one of the two remaining free points of S is to the left l' and the other one is to its right. Then, map s_2 to the lowest of such two points, map t_1 to the highest of such two points, and map s_1 to $b(S \setminus S_4)$.

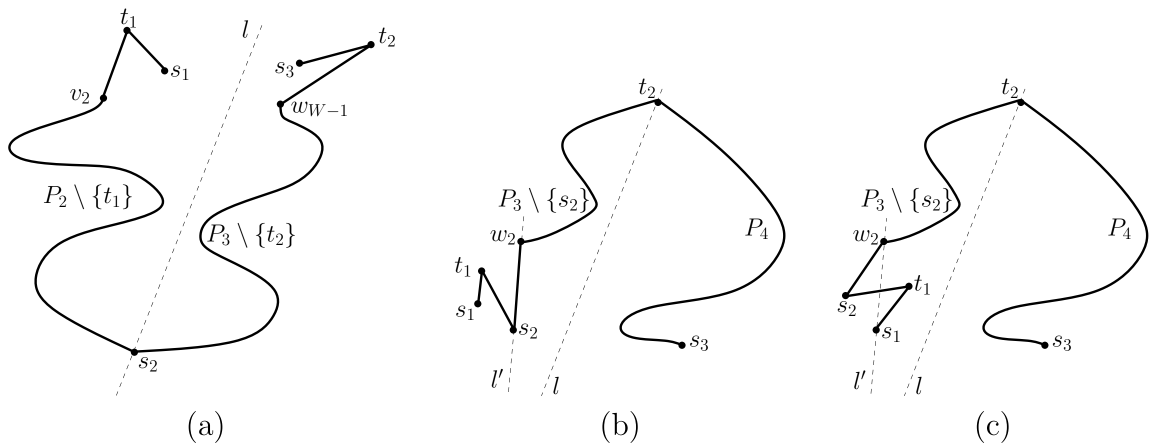

Figure 12: (a) Embedding a path with five switches whose first and fourth monotone paths are trivial into a general point set. (b)–(c) Embedding a path with five switches whose first and second monotone paths are trivial into a general point set.

Claim 7 *The constructed straight-line embedding of P into S is upward and planar.*

Proof: Refer to Figs. 12(b)-(c). The constructed straight-line embedding is upward as P_1, P_2, P_3, and P_4 are drawn upward by construction. Further, it is planar, namely the drawing of P_4 lies entirely to the right of l, while the drawings of P_1, P_2, and P_3 lie entirely to the left of l; further, (s_1, t_1) and (s_2, t_1) do not cross as they are adjacent; moreover, the only edge of P_3 that crosses the smallest horizontal strip containing (s_1, t_1) and (s_2, t_1) is (w_1, w_2); however, such an edge does not cross with (s_2, t_1) as they are adjacent and does not cross with (s_1, t_1) as such two edges lie one to the left and one to the right of l' (if one of the two remaining free points of S is to the left of l' and the other one is to the right) or lie one on l' and one to the left or to the right of l' (if both the two remaining free points of S are on the same side of l'). □

We discuss the more involved case in which P_2 and P_3 are trivial. Let p_1 and p_2 be the two highest points of S, with $y(p_1) \geq y(p_2)$. Counterclockwise separate a set S_1 of $U - 1$ points around p_2. Denote by p the point that is added to S_1 when U points are counterclockwise separated around p_2. We consider the following two cases:

Point p is to the left of $l_{1,2}$: Refer to Fig. 13(a). Consider a half-line l_1 fixed at p_1 and passing through p. Rotate l_1 in counterclockwise direction. Let p' be the last point of S_1 encountered by l_1 before encountering p_2. If no point of S_1 is encountered by l_1 before p_2, then let $p' = p$. Map t_1 to p_1, t_2 to p_2, and s_2 to p'; construct upward straight-line embeddings of P_1 into $S_1 \cup \{p_1, p\} \setminus \{p'\}$ and of P_4 into $S \setminus \{S_1\} \cup \{p_2\} \setminus \{p\}$.

Point p is to the right of $l_{1,2}$: Refer to Fig. 13(b). Consider a half-line l_1 fixed at p_1 and passing through p. Rotate l_1 in clockwise direction. Let p' be the last point of S_1 encountered by l_1 before encountering p_2. If no point of S_1 is encountered by l_1 before p_2, then let $p' = p$. Map t_1 to p_2, t_2 to p_1, and s_2 to p'; construct upward straight-line embeddings of P_1 into $S_1 \cup \{p_2\}$ and of P_4 into $S \setminus \{S_1\} \cup \{p_1, p\} \setminus \{p'\}$.

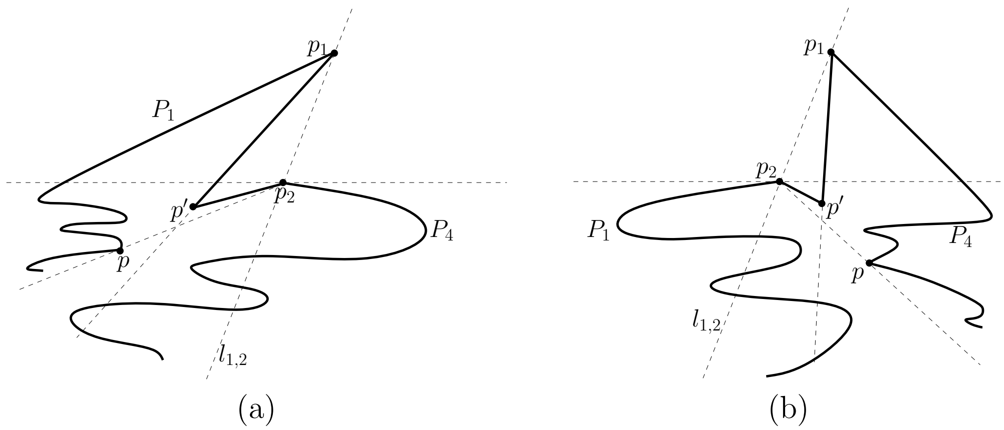

Figure 13: (a) Point p is to the left of $l_{1,2}$. (b) Point p is to the right of $l_{1,2}$.

Claim 8 *The constructed straight-line embedding of P into S is upward and planar.*

Proof: Suppose first that p is to the left of $l_{1,2}$. We prove the upwardness of the constructed embedding. By construction, $y(p') < y(p_2)$ and $y(p') < y(p_1)$ hold. Hence, edges (s_2, t_1) and (s_2, t_2) are drawn upward. Further, paths P_1 and P_4 are drawn upward by construction. We now prove the planarity. Path P_1 is entirely contained in the convex region delimited by the line through p_1 and p', and by the line through p_2 and p; as no other edge of P intersects the interior of such a half-plane, the edges of P_1 do not cross any other edge of P. Path P_4 is entirely contained in the convex region delimited by the horizontal line through p_2 and by the line through p_2 and p; as no other edge of P intersects the interior of such a convex region, the edges of P_4 do not cross any other edge of P. Finally, edges (s_2, t_1) and (s_2, t_2) are adjacent, hence they do not cross.

Suppose now that p is to the right of $l_{1,2}$. We prove the upwardness of the constructed embedding. By construction, $y(p') < y(p_2)$ and $y(p') < y(p_1)$ hold. Hence, edges (s_2, t_1) and (s_2, t_2) are drawn upward. Further, paths P_1 and P_4 are drawn upward by construction. We now prove the planarity. Path P_1 is entirely contained in the convex region delimited by the line through p_2 and p and by the horizontal line through p_2; as no other edge of P intersects the interior of such a convex region, the edges of P_1 do not cross any other edge of P. Path P_4 is entirely contained in the convex region delimited by line through p_1 and p' and by the line through p_2 and p; as no other edge of P intersects the interior of such a convex region, the edges of P_4 do not cross any other edge of P. Finally, edges (s_2, t_1) and (s_2, t_2) are adjacent, hence they do not cross. □

□

Next, we tackle the problem of embedding paths with at most k switches into general point sets with more than n points. We show the following:

Theorem 6 *Every directed path P with n vertices and k switches admits an upward straight-line embedding into every general point set S with $|S| \geq n2^{k-2}$.*

Proof: We prove the statement by induction on the number of switches; we suppose inductively that one of the end-vertices of P is mapped to $b(S)$ or $t(S)$, depending on whether such a vertex is a source or a sink. The statement is trivial if $k = 2$, as in such a case P is monotone and any general point set with n points suffices.

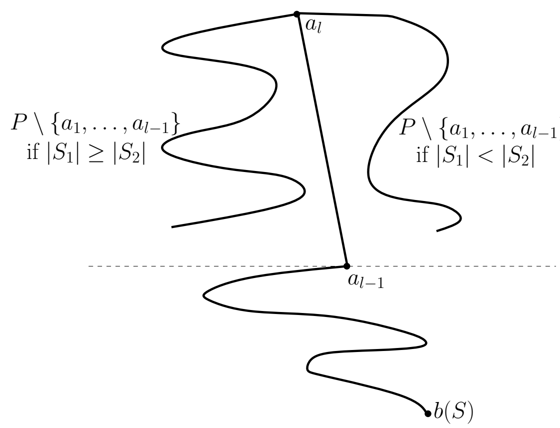

Figure 14: Illustration for the proof of Theorem 6.

Suppose that $k > 2$. Refer to Fig. 14. Let a_1 be an end-vertex of P. Suppose that a_1 is a source, the case in which it is a sink being analogous. Let $P_a = (a_1, a_2, \ldots, a_l)$ be the maximal monotone path of P containing a_1. Notice that $l \geq 2$. Map $P_a \setminus \{a_l\}$ to the $|P_a| - 1$ points of S with lowest y-coordinate. Denote such a point set by S_a. Map a_l to $t(S)$. Let S_1 and S_2 be the point sets composed of $t(S)$ and of the points of $S \setminus S_a$ to the left and to the right, respectively, of the line through a_{l-1} and a_l. If $|S_1| \geq |S_2|$ (if $|S_2| > |S_1|$), construct an upward straight-line embedding of $P \setminus \{a_1, a_2, \ldots, a_{l-1}\}$ into S_1 (into S_2, resp.) with a_l placed at $t(S_1) = t(S)$ (at $t(S_2) = t(S)$, resp.).

It is easy to see that the constructed straight-line embedding is upward and planar. We show that the cardinality of point sets S_1 and S_2 is sufficient to apply the induction. The number of points in the one of S_1 and S_2 with more points is at least $(|S| - (l-1))/2$. Further, $P \setminus \{a_1, a_2, \ldots, a_{l-1}\}$ has $n - (l-1)$ vertices and $k - 1$ switches. Since $|S| \geq n2^{k-2}$, the one of S_1 and S_2 with more points has at least $(n2^{k-2} - (l-1))/2 = n2^{k-3} - (l-1)/2 > n2^{k-3} - (l-1)2^{k-3}$ and, since $k > 2$, the lemma follows. □

6 Open Problems

In this paper we continued the study of upward straight-line embeddability of directed graphs into point sets initiated in [8, 3]. While we solved some of the open questions posed by Binucci *et al.* in [3], the following problems remain open:

- Is it possible to test in polynomial time whether a directed graph/tree admits an upward straight-line embedding into every point set in general/convex position?
- Does every directed path admit an upward straight-line embedding into every point set in general position?
- Is there a polynomial function $p(n, k)$ such that every directed path admits an upward straight-line embedding into every point set in general position with at least $p(n, k)$ points? Theorem 6 shows that every directed path admits an upward straight-line embedding into every point set in general position with at least $n2^{k-2}$ points, which is exponential in k.

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