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On a Tree and a Path with no Geometric Simultaneous Embedding

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ABSTRACT

Two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ admit a geometric simultaneous embedding if there exists a set of points P and a bijection $M : P \to V$ that induce planar straight-line embeddings both for G_1 and for G_2. The most prominent problem in this area is the question whether a tree and a path can always be simultaneously embedded. We answer this question in the negative by providing a counterexample. Additionally, since the counterexample uses disjoint edge sets for the two graphs, we also negatively answer another open question, that is, whether it is possible to simultaneously embed two edge-disjoint trees. Finally, we study the same problem when some constraints on the tree are imposed. Namely, we show that a tree of height 2 and a path always admit a geometric simultaneous embedding. In fact, such a strong constraint is not so far from closing the gap with the instances not admitting any solution, as the tree used in our counterexample has height 4.

1 Introduction

Embedding planar graphs is a well-established field in graph theory and algorithms with a great variety of applications. Keystones in this field are the works of Thomassen [17], of Tutte [18], and of Pach and Wenger [16], dealing with planar and convex representations of graphs in the plane.

Recently, motivated by the need of concurrently represent several different relationships among the same set of elements, a major focus in the research lies on *simultaneous graph embedding*. In this setting, given a set of graphs with the same vertex-set, the goal is to find a set of points in the plane and a mapping between these points and the vertices of the graphs that yields a planar embedding for both the graphs, when displayed separately. Problems of this kind frequently arise when dealing with the visualization of evolving networks and with the visualization of huge and complex relationships, as the graph of the Web.

Among the many variants of this problem, the most important and natural one is the *geometric simultaneous embedding*. Given two graphs $G_1 = (V, E')$ and $G_2 = (V, E'')$, the task is to find a set of points P and a bijection $M : P \to V$ that induce planar *straight-line* embeddings for both G_1 and G_2.

In the seminal paper on this topic [2], Brass *et al.* proved that geometric simultaneous embeddings of pairs of paths, pairs of cycles, and pairs of caterpillars always exist. A *caterpillar* is a tree such that deleting all its leaves yields a path. On the other hand, many negative results have been shown. Brass *et al.* [2] presented a pair of outerplanar graphs not admitting any geometric simultaneous embedding and provided negative results for three paths, as well. Erten and Kobourov [5] found a planar graph and a path not allowing any geometric simultaneous embedding. Geyer *et al.* [13] proved that there exist two edge-disjoint trees that do not admit any geometric simultaneous embedding. Finally, Cabello *et al.* [3] showed a planar graph and a matching that do not admit any geometric simultaneous embedding and presented algorithms to obtain a geometric simultaneous embedding of a matching and a wheel, an outerpath, or a tree. The most important open problem in this area is the question whether a tree and a path always admit a geometric simultaneous embedding or not, that is the subject of this paper.

Many variants of the problem, where some constraints are relaxed, have been studied. In the *simultaneous embedding* setting, where the edges do not need to be straight-line segments, any number of planar graphs admit a simultaneous embedding, since any planar graph can be planarly embedded on any given set of points in the plane [15, 16]. However, the same result does not hold if the edges that are shared by the two graphs have to be represented by the same Jordan curve. In this setting the problem is called *simultaneous embedding with fixed edges* [10, 12, 7]. Finally, the research on this problem opened a new exciting field of problems and techniques, like ULP trees and graphs [6, 8, 9], colored simultaneous embedding [1], near-simultaneous embedding [11], and matched drawings [4], deeply related to the general fundamental question of point-set embeddability.

In this paper we study the geometric simultaneous embedding problem of a tree and a path. We answer the question in the negative by providing a counterexample, that is, a tree and a path not admitting any geometric simultaneous embedding. Moreover, since the tree and the path used in our counterexample do not share any edge, we also negatively answer the question on two edge-disjoint trees.

The main idea behind our counterexample is to use the path to enforce a part of the tree to be in a certain configuration which cannot be drawn planar. Namely, we make use of level nonplanar trees [6, 9], that is, trees not admitting any planar embedding if their vertices have to be placed inside certain regions according to a particular leveling. The tree of the counterexample contains many copies of such trees, while the path is used to create the regions. To prove that at least one copy has to be in the particular leveling that determines a crossing, we need a quite huge number of vertices. However, such a huge number is often needed just to ensure the existence of particular structures playing a role in our proof. A much smaller counterexample could likely be constructed with the same techniques, but we decided to prefer the simplicity of the argumentations rather than the search for the minimum size.

The paper is organized as follows. In Sect. 2 we give preliminary definitions and we introduce the concept of level nonplanar trees. In Sect. 3 we describe the tree $\mathcal{T}$ and the path $\mathcal{P}$ used in the counterexample. In Sect. 4 we give an overview of the proof that $\mathcal{T}$ and $\mathcal{P}$ do not admit any geometric simultaneous embedding, while in Sect. 5 we give the details of such a proof. In Sect. 6 we give an algorithm to construct a geometric simultaneous embedding of a tree of height 2 and a path and in Sect. 7 we make some final remarks.

2 Preliminaries

A (undirected) *k-level tree* $T = (V, E, \phi)$ is a tree $T' = (V, E)$, called the *underlying tree* of T, together with a leveling of its vertices given by a function $\phi : V \mapsto \{1, \dots, k\}$, such that for every edge $(u, v) \in E$, it holds $\phi(u) \neq \phi(v)$ (See [6, 9]). A drawing of $T = (V, E, \phi)$ is a *level drawing* if each vertex $v \in V$ such that $\phi(v) = i$ is placed on a horizontal line $l_i = \{(x, i) \mid x \in \mathbb{R}\}$. A level drawing of T is *planar* if no two edges intersect except, possibly, at common end-points. A tree $T = (V, E, \phi)$ is *level nonplanar* if it does not admit any planar level drawing.

We extend this concept to the one of *region-level drawing* by enforcing the vertices of each level to lie inside a certain region rather than on a horizontal line. Let $l_1, \dots, l_k$ be k non-crossing straight-line segments and let $r_1, \dots, r_{k+1}$ be the regions of the plane such that any straight-line segment connecting a point in r_i and a point in r_h, with $1 \leq i < h \leq k + 1$, cuts all and only the segments $l_i, l_{i+1}, \dots, l_{h-1}$, in this order. A drawing of a k-level tree $T = (V, E, \phi)$ is called *region-level drawing* if each vertex $v \in V$ such that $\phi(v) = i$ is placed inside region r_i. A tree $T = (V, E, \phi)$ is *region-level nonplanar* if it does not admit any planar region-level drawing. The 4-level tree T whose underlying tree is shown in Fig. 1(a) is level nonplanar [9] (see Fig. 1(b)). We show that T is also region-level nonplanar.

Lemma 1 *The 4-level tree T whose underlying tree is shown in Fig. 1(a) is region-level nonplanar.*

Proof: Refer to Fig. 1(c). First observe that, in any possible region-level planar drawing of T, there exists a polygon Q_2 inside region r_2 delimited by paths $p_1 = v_5, v_2, v_8$ and $p_2 = v_6, v_3, v_9$, and by segments l_1 and l_2, and a polygon Q_3 inside region r_3 delimited by paths p_1 and p_2, and by segments l_2 and l_3. We have that v_1 is inside Q_2, as otherwise one of edges (v_1, v_2) or (v_1, v_3) would cross one of p_1 or p_2. Hence, vertex v_4 has to be inside Q_3, as otherwise edge (v_1, v_4) would cross one of p_1 or p_2. However, in this case,

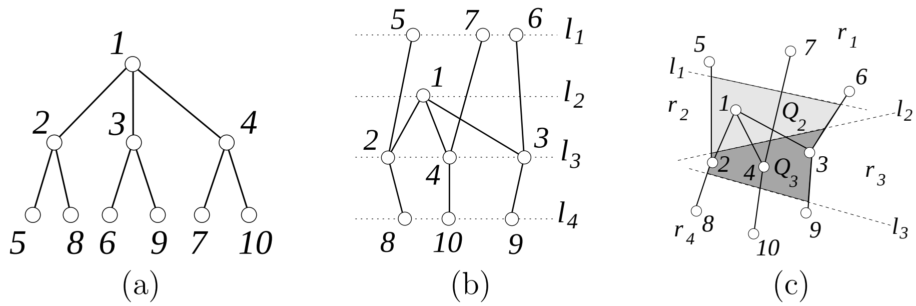


Figure 1: (a) A tree T_u. (b) A level nonplanar tree T whose underlying tree is T_u. (c) A region-level nonplanar tree T whose underlying tree is T_u.

there is no placement for vertices v_7 and v_{10} that avoids a crossing between one of edges (v_4, v_7) or (v_4, v_{10}) and one of the already drawn edges. □

Lemma 1 will be vital for proving that there exist a tree $\mathcal{T}$ and a path $\mathcal{P}$ not admitting any geometric simultaneous embedding. In fact, $\mathcal{T}$ contains many copies of the underlying tree of T, while $\mathcal{P}$ connects vertices of $\mathcal{T}$ in such a way to create the regions satisfying the above conditions and to enforce at least one of such copies to lie inside these regions according to the leveling that makes it nonplanar.

3 The Counterexample

In this section we describe a tree $\mathcal{T}$ and a path $\mathcal{P}$ not admitting any geometric simultaneous embedding.

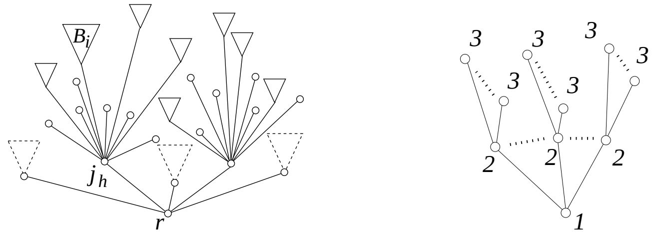
3.1 Tree $\mathcal{T}$

The tree $\mathcal{T}$ contains a root r and q vertices $j_1, \ldots, j_q$ at distance 1 from r, called *joints*. Each joint j_h, with $h = 1, \ldots, q$, is connected to $l := (s-1)^4 \cdot 3^2 \cdot x$ vertices of degree 1, called *stabilizers* and to x subtrees B_i, $i = 1, \ldots, x$, called *branches*, each one consisting of a root r_i, $(s-1) \cdot 3$ vertices of degree $(s-1)$ adjacent to r_i, and $(s-2) \cdot (s-1) \cdot 3$ leaves at distance 2 from r_i. See Fig. 2(a) for a schematization of $\mathcal{T}$ and Fig. 2(b) for a schematization of a branch. Vertices belonging to a branch B_i are called *B-vertices* and denoted by 1-, 2-, or 3-vertices, according to their distance from their joint.

Because of the huge number of vertices, in the rest of the paper, for the sake of readability, we use variables q, s, and x as parameters describing the size of certain structures. Such parameters will be given a value when the technical details are described. At this stage we just claim that a total number $n \geq \binom{2^7 \cdot 3 \cdot x + 2}{3}$ of vertices (see Lemmata 5 and 4) suffices for the counterexample.

As a first observation we note that, despite the oversized number of vertices, tree $\mathcal{T}$ has limited *height*, that is, every vertex is at distance from the root at most 4. This leads to the following property.

Property 1 *Any simple path of tree-edges starting at the root has at most 3 bends.*



(a) (b)

Figure 2: (a) A schematization of $\mathcal{T}$. Joints and stabilizers are small circles. A solid triangle represents a branch, while a dashed triangle represents the subtree connected to a joint. (b) A schematization of a branch B_i. Vertices are labeled with their distance from the joint the branch is connected to.

3.2 Path $\mathcal{P}$

Path $\mathcal{P}$ is given by describing some basic and recurring subpaths on the vertices of $\mathcal{T}$ and how such subpaths are connected to each other. The idea is to partition the set of branches adjacent to each joint j_h into subsets of s branches each and to connect the vertices of each set with path-edges, according to some features of the tree structure, so defining the first building block, called *cell*. Then, cells belonging to the same joint are connected to each other to create structures, called *formations*, for which we can ensure certain properties regarding the intersection between tree and path-edges. Further, different formations are connected to each other by path-edges in such a way to create bigger structures, called *extended formations*, which, in their turn, are connected to create a *sequence of extended formations*.

All of these structures are constructed in such a way that there exists a set of cells, connected to the same joint and being part of the same formation or extended formation, such that any four of these cells contain a copy of a region-level nonplanar tree, where the level of a vertex is determined by the cell it belongs to. Hence, proving that four of such cells lie in different regions satisfying the properties of separation described above is equivalent to proving the existence of a crossing in $\mathcal{T}$. This allows us to consider only the bigger structures instead of dealing with single copies of the region-level nonplanar tree.

In the following we define such structures more formally and state their properties.

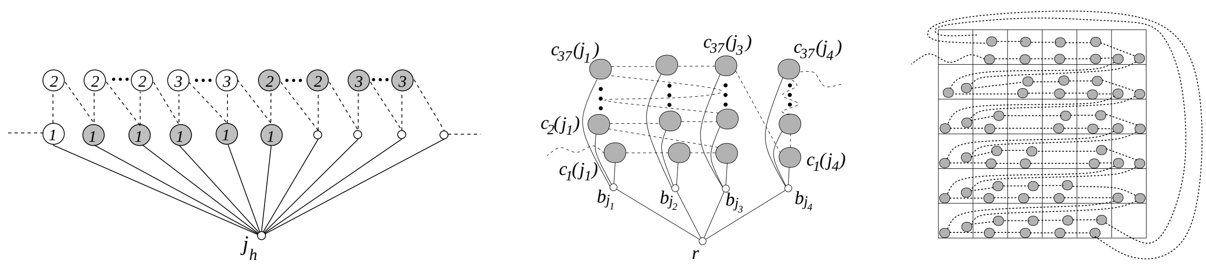
Cell: The most basic structure is defined by looking at how $\mathcal{P}$ connects the vertices of a set of s branches connected to the same joint of $\mathcal{T}$. Assume the vertices of a level inside each branch to be arbitrarily ordered.

For each joint j_h, $h = 1, \dots, q$, and for each disjoint subset of s branches B_i, $i = 1, \dots, s$, connected to j_h, we construct a set of s cells as follows. For each $i = 1, \dots, s$, a *cell* $c_i(h)$ is composed of its *head*, its *tail*, and a number t of stabilizers. The *head* of $c_i(h)$ consists of the unique 1-vertex of B_i, the first three 2-vertices of each branch B_k, with $1 \leq k \leq s$ and $k \neq i$, that are not already used in a cell $c_a(h)$ with $1 \leq a < r$ and, for each 2-vertex not in $c_i(h)$ and not in B_i, the first 3-vertices not already used in a cell $c_a(h)$, with $1 \leq a < i$. The *tail* of $c_i(h)$ is created by considering a set of $3 \cdot s \cdot (s-1)^2$ branches adjacent to j_h, partitioned into $3 \cdot (s-1)^2$ subsets of s subtrees each. The vertices of each subset are distributed between the cells in the same way as for the vertices of the head.

This implies that each cell contains one 1-vertex, $3\cdot(s-1)$ 2-vertices, and $3\cdot(s-2)\cdot(s-1)$ 3-vertices of the head, an additional $3\cdot(s-1)^2$ 1-vertices, $3^2\cdot(s-1)^3$ 2-vertices, and $3^2\cdot(s-2)\cdot(s-1)^3$ 3-vertices of the tail, plus $3^2\cdot(s-1)^4$ stabilizers.

Path $\mathcal{P}$ inside cell $c_i(h)$ visits the vertices in the following order: It starts at the unique 1-vertex of the head, then it reaches all the 2-vertices of the head, then all the 3-vertices of the head, then all the 2-vertices of the tail, and finally all the 3-vertices of the tail, visiting each set in arbitrary order. After each occurrence of a 2- or 3-vertex of the head, $\mathcal{P}$ visits a 1-vertex of the tail, and after each occurrence of a 2- or a 3-vertex of the tail, it visits a stabilizer of joint j_h (see Fig. 3(a)).

Note that each set of s cells constructed starting from the same set of s branches is such that each subset of size four contains a region-level nonplanar tree, where the levels correspond to the membership of the vertices to a cell. Namely, consider four cells $c_1,\ldots,c_4$ belonging to the same set, leveled in this order. A region level nonplanar tree as in Fig. 1(c) consists of the 1-vertex v of the head of c_2, the three 2-vertices of c_3 connected to v and, for each of them, the 3-vertex of c_1 and the 3-vertex of c_4 connected to it.



(a) (b) (c)

Figure 3: (a) A cell. Vertices of the head are white and vertices of the tail are grey. B-vertices are large and stabilizers are small circles. (b) A formation. (c) A subsequence $(H_1,\ldots,H_x)^2$ of an extended formation. Formations are inside a table to represent the 4-tuple they belong to and to emphasize that in each repetition (a row of the table) a formation at a certain 4-tuple is missing.

Formation: In the definition of cells we described how the path traverses through one set of branches connected to the same joint. Now we describe how cells from four different sets are connected to each other.

A *formation* $F(H)$, where $H=(h_1,h_2,h_3,h_4)$ is a 4-tuple of indices of joints, consists of 592 cells. Namely, for each joint j_{h_i}, $1\le i\le 4$, $F(H)$ contains 148 cells belonging to the same set of s cells connected to j_{h_i}. Path $\mathcal{P}$ connects these cells in the order $((h_1h_2h_3)^{37}h_4^{37})^4$, that is, $\mathcal{P}$ repeats four times the following sequence: It connects $c_1(h_1)$ to $c_1(h_2)$, then to $c_1(h_3)$, then to $c_2(h_1)$, and so on till $c_{37}(h_3)$, from which it then connects to $c_1(h_4)$, to $c_2(h_4)$, and so on till $c_{37}(h_4)$ (see Fig. 3(b)). A connection between two consecutive cells $c_r(a)$ and $c_r(b)$ is done with an edge between the end vertices of the subpaths $\mathcal{P}$ $(c_r(a))$ and $\mathcal{P}$ $(c_r(b))$ of $\mathcal{P}$ induced by the vertices of $c_r(a)$ and $c_r(b)$, respectively. Namely, the unique vertex in $c_r(a)$ having degree 1 both in $\mathcal{P}$ $(c_r(a))$ and in $\mathcal{T}$ is connected to the unique vertex in $c_r(b)$ having degree 1 in $\mathcal{P}$ $(c_r(b))$ but not in $\mathcal{T}$. Since, by construction, the cells of $F(H)$ that are connected to the same joint belong to the same set of s cells, the following property holds:

Property 2 *For any formation $F(H)$ and any joint j_h, with $h\in H$, if four cells $c_r(h)\in F(H)$ are pairwise separated by straight lines, then there exists a crossing in $\mathcal{T}$.*

Extended Formation: Formations are connected by the path in a special sequence, called *extended formation* and denoted by $EF(H)$, where $H = (H_1 = (h_1, \ldots, h_4), H_2 = (h_5, \ldots, h_8), \ldots, H_x = (h_{4x-3}, \ldots h_{4x}))$ is an x-tuple of $4-$tuples of disjoint indexes of joints. For each $4-$tuple H_i, $EF(H)$ contains $y - \frac{y}{x}$ formations $F_1(H_i), \ldots, F_{y-\frac{y}{x}}(H_i)$ not belonging to any other extended formation and composed of cells of the same set of s cells connected to the same joint (see Fig. 3(c)). Formations inside $EF(H)$ are connected in $\mathcal{P}$ in the order $(H_1, H_2, \ldots, H_x)^y$, that is, $\mathcal{P}$ connects $F_1(H_1)$ to $F_1(H_2)$, then to $F_1(H_3)$, and so on till $F_1(H_x)$, then to $F_2(H_1)$, to $F_2(H_2)$, and so on till $F_{y-\frac{y}{x}}(H_x)$. However, in each of these y repetitions one H_i is missing. Namely, in the k-th repetition the path does not reach any formation at H_m, with $m = k \mod x$. We say that the k-th repetition has a *defect* at m. We call a subsequence $(H_1, H_2, \ldots, H_x)^x$ a *full repetition*. Observe that a full repetition has exactly one defect at each tuple.

Note that the size of s can now be fixed as the number of formations creating repetitions inside one extended formation times the number of cells inside each of these formations, that is, $s := (y - \frac{y}{x}) \cdot 37 \cdot 4$. We claim that $x = 7 \cdot 3^2 \cdot 2^{23}$ and $y = 7^2 \cdot 3^3 \cdot 2^{26}$ is sufficient throughout the proofs. However, for readability reasons, we will keep on using variables x and y in the remainder of the paper.

Sequence of Extended Formations: Extended formations are connected by the path in a special sequence, called *sequence of extended formations* and denoted by $SEF(H)$, where $H = (H_1^*, \ldots, H_{12}^*)$ is a $12-$tuple of x-tuples of $4-$tuples of disjoint indexes of joints. For each x-tuple H_i^*, with $i = 1, \ldots, 12$, consider 110 extended formations $EF_j(H_i^*)$, with $j = 1, \ldots, 110$, not already belonging to any other sequence of extended formations. These extended formations are connected by $\mathcal{P}$ in the order $(H_1^*, \ldots, H_{12}^*)^{(120)}$, that is, $\mathcal{P}$ connects $EF_1(H_1^*)$ to $EF_1(H_2^*)$, then to $EF_1(H_3^*)$, and so on till $EF_1(H_{12}^*)$, then to $EF_2(H_1^*)$, to $EF_2(H_2^*)$, and so on till $EF_{110}(H_{12}^*)$. There exist two types of sequences of extended formations. Namely, in the first type, in each repetition $(H_1^*, \ldots, H_{12}^*)$ there is one extended formation $EF(H_m)$ missing. In this case, as for the extended formations, we say that the repetition has a *defect* at m. In the second type, in each repetition $(H_1^*, \ldots, H_{12}^*)$ two consecutive extended formations are missing. Namely, in the k-th repetition the path skips the extended formations $EF(H_m^*)$ and $EF(H_{m+1}^*)$, with $m = k \bmod 12$. In this case, we say that the repetition has a *double defect* at m.

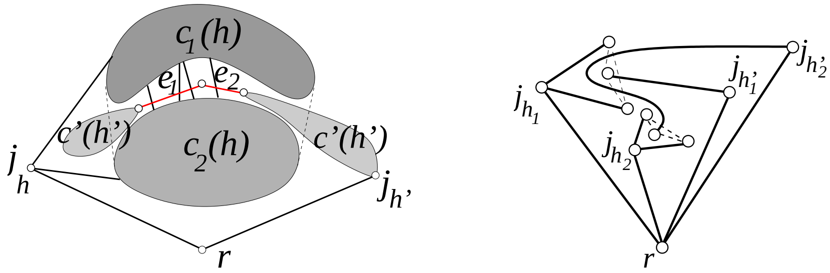
Then, we can fix $q = 48x$, as we need 4 sequences of extended formations (of size 12 each) not sharing any joint. As, for each set of $48x$ joints, $(48x)!$ different disjoint sequences of extended formations exist, we just consider the sequences where the order of the tuple is the order of the joints around the root.

4 Overview

In this section we present the main arguments leading to the final conclusion that the tree $\mathcal{T}$ and the path $\mathcal{P}$ described in Sect. 3 do not admit any geometric simultaneous embedding. For the sake of readability, we decided to give the outline of the proof in this section and to defer some of the longest proofs to Sect. 5.

The main idea in this proof scheme is to use the structures given by the path to fix a part of the tree in a specific shape creating restrictions for the placement of the further substructures of $\mathcal{T}$ and of $\mathcal{P}$ attached to it. Then, we show that such restrictions lead to a crossing in any possible drawing of $\mathcal{P}$ and $\mathcal{T}$. In the following, we will perform an analysis of the geometrical properties of all possible embeddings in order to show that none of them is feasible. Hence, throughout the proof, we will assume the graph as embedded in a certain way and we will show that such an embedding determines a crossing.

We first give some further definitions and basic topological properties on the interaction among cells that are enforced by the preliminary arguments about region-level planar drawings and by the order in which the subtrees are connected inside one formation.



(a) (b)

Figure 4: (a) A passage between cells c_1, c_2, and c'. (b) Two interconnected passages.

Passage: Consider two cells $c_1(h), c_2(h)$ connected to a joint j_h that can not be separated by a straight line and a cell $c'(h')$ connected to a joint $j_{h'}$, with $h' \neq h$. We say that there exists a *passage* P between c_1, c_2, and c' if the polyline given by the path of c' separates vertices of c_1 from vertices of c_2 (see Fig. 4(a)). Observe that, since the polyline can not be straight, there is a vertex of c' lying inside the convex hull of the vertices of $c_1 \cup c_2$. Hence, there exist at least two path-edges e_1, e_2 of c' that are intersected by tree-edges connecting vertices of c_1 to vertices of c_2.

For two passages P_1 between $c_1(h_1)$, $c_2(h_1)$, and $c'(h'_1)$, and P_2 between $c_3(h_2)$, $c_4(h_2)$, and $c'(h'_2)$ (without loss of generality, assume $h_1 < h'_1$, $h_2 < h'_2$, and $h_1 < h_2$), we distinguish three different configurations: (i) If $h'_1 < h_2$, P_1 and P_2 are *independent*; (ii) if $h'_2 < h'_1$, P_2 is *nested* into P_1; and (iii) if $h_2 < h'_1 < h'_2$, P_1 and P_2 are *interconnected* (see Fig. 4(b)).

Doors: Let $c_1(h), c_2(h)$, and $c'(h')$ be three cells creating a passage. Consider any triangle given by a vertex v' of c' inside the convex hull of $c_1 \cup c_2$ and by any two vertices of $c_1 \cup c_2$. This triangle is a *door* if it encloses neither any other vertex of c_1, c_2 nor any vertex of c' that is closer than v' to $j_{h'}$ in $\mathcal{T}$. A door is *open* if no tree-edge incident to v' crosses the opposite side of the triangle, that is, the side between the vertices of c_1 and c_2 (see Fig. 5(a)), otherwise it is *closed* (see Fig. 5(b)).

Consider two joints j_a and j_b, with j_h, j_a, j'_h, j_b appearing in this circular order around the root. Any polyline connecting the root to j_a, then to j_b, and again to the root, without crossing tree-edges, must traverse each door by crossing both the sides adjacent to v'. If a door is closed, such a polyline has to bend after crossing one side adjacent to v' and before crossing the other one.

Observe that, if two passages P_1 and P_2 are interconnected, then either all the closed doors of P_1 are traversed by a path of tree-edges belonging to P_2 or all the closed doors of P_2 are traversed by a path of tree-edges belonging to P_1 (see Fig. 4(b)).

In the rest of the argument we will exploit the fact that the closed door of a passage requires a bend in the tree to obtain the claimed property that a large part of $\mathcal{T}$ has to follow the same shape. In view of this, we state the following lemmata relating the concepts of doors, passages, and formations.

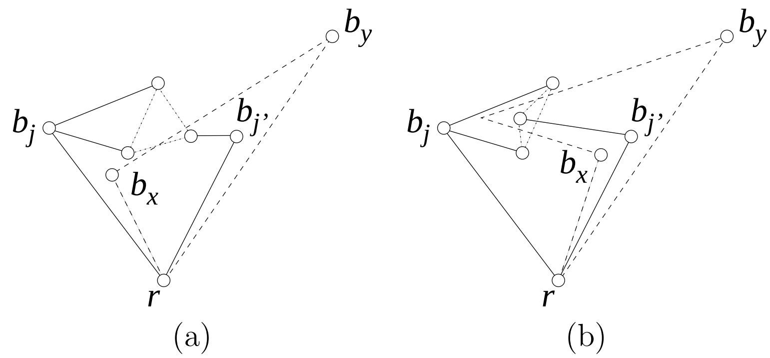


Figure 5: (a) An open door. (B) A closed door.

Lemma 2 *For each formation $F(H)$, with $H = (h_1, \ldots, h_4)$, there exists a passage between some cells $c_1(h_a), c_2(h_a), c'(h_b) \in F(H)$, with $1 \le a, b \le 4$.*

Lemma 3 *Each passage contains at least one closed door.*

Hence, each formation contains at least one closed door. In the following we prove that the effects of closed doors belonging to different formations can be combined to obtain more restrictions on the shape of the tree. First, we exploit a combinatorial argument based on the Ramsey Theorem [14] to state that there exists a set of joints such that any two joints in this set contain cells creating a passage.

Lemma 4 *Given a set of joints $J = \{j_1, \ldots, j_y\}$, with $|J| = y := \binom{2^7 \cdot 3 \cdot x + 2}{3}$, there exists a subset $J' = \{j'_1, \ldots, j'_r\}$, with $|J'| = r \ge 2^7 \cdot 3 \cdot x$, such that for each pair of joints $j'_i, j'_h \in J'$ there exist two cells $c_1(i), c_2(i)$ creating a passage with a cell $c'(h)$.*

Then, we give some further definitions concerning the possible shapes of the tree.

Enclosing bendpoints: Consider two paths $p_1 = \{u_1, v_1, w_1\}$ and $p_2 = \{u_2, v_2, w_2\}$. The bendpoint v_1 of p_1 *encloses* the bendpoint v_2 of p_2 if v_2 is internal to triangle $\triangle(u_1, v_1, w_1)$. See Fig. 6(a).

Channels: Consider a set of joints $J = \{j_1, \ldots, j_k\}$ in clockwise order around the root. The *channel* c_i of a joint j_i, with $i = 2, \ldots, k-1$, is the region given by the pair of paths, one path of j_{i-1} and one path of j_{i+1}, with the maximum number of enclosing bendpoints with each other. We say that c_i is an *x-channel* if the number of enclosing bendpoints is x. Observe that, by Prop. 1, $x \le 3$. A 3-channel is depicted in Fig. 6(b). Note that, given an x-channel c_i of j_i, all the vertices of the subtree rooted at j_i that are at distance at most x from the root lie inside c_i.

Channel segments: An x-channel c_i is composed of $x+1$ *channel segments.* The first channel segment cs_1 is the part of c_i that is visible from the root. The h-th channel segment cs_h is the part of c_i disjoint from cs_{h-1} that is bounded by the elongations of the paths of j_{i-1} and j_{i+1} after the h-th bend. The *bending area* $b(a, a+1)$ of c_i is the region that is visible from all the points of channel segments cs_a and cs_{a+1}.

Observe that, as the channels are created by tree-edges, any tree-edge connecting vertices inside the channel has to be drawn inside the channel, while path-edges can cross the boundaries of the channel, hence possibly crossing other channels. We study the relationships between path-edges and channels.

The following property descends from the fact that, by construction, every second vertex reached by $\mathcal{P}$ in a cell is either a 1-vertex or a stabilizer.

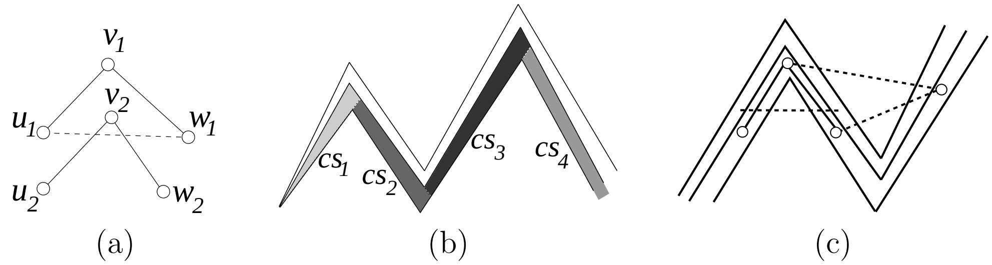


Figure 6: (a) An enclosing bendpoint. (b) A 3-channel and its channel segments. (c) A blocking cut.

Property 3 *For any path-edge $e = (a, b)$, at least one of a and b lie inside either cs_1 or cs_2.*

Blocking cuts: A *blocking cut* is a path-edge connecting two consecutive channel segments by cutting some of the other channels twice. See Fig. 6(c).

Property 4 *Let c be a channel that is cut twice by a blocking cut. If c has vertices in both the channel segments cut by the path-edge, then it has some vertices in a different channel segment.*

Now we are ready to prove that most of the joints have to create the same shape. First, based on Prop. 4, we show that any set of joints as in Lemma 4 contains a particular subset, composed of joints creating interconnected passages, such that each pair of paths of tree-edges starting at the root and containing such joints has at least two common enclosing bendpoints, which implies that most of them create 2-channels.

From now on, we identify a joint with the channel it belongs to. Then, when dealing with a passage between two cells $c_1(h), c_2(h)$ of a joint j_h and a cell $c'(h')$ of a joint $j_{h'}$, we might also say that there is a passage between joints j_h and j'_h or between the corresponding channels.

Lemma 5 *Consider a set of joints $J = \{j_1, \ldots, j_k\}$ such that there exists a passage between each pair (j_i, j_h), with $1 \leq i, h \leq k$. Let $\mathcal{P}_1 = \{P \mid P$ connects c_i and $c_{\frac{3k}{4}+1-i}$, for $i = 1, \ldots, \frac{k}{4}\}$ and $\mathcal{P}_2 = \{P \mid P$ connects $c_{\frac{k}{4}+i}$ and c_{k+1-i}, for $i = 1, \ldots, \frac{k}{4}\}$ be two sets of passages between pairs of joints in J (see Fig. 7). Then, for at least $\frac{k}{4}$ of the joints of one set of passages, say $\mathcal{P}_1$, there exist paths in $\mathcal{T}$, starting at the root and containing these joints, which traverse all the doors of $\mathcal{P}_2$ with at least 2 and at most 3 bends. Also, at least half of these joints create an x-channel, with $2 \leq x \leq 3$.*

By Lemma 5, any formation attached to a certain subset of joints must use at least three different channel segments. In the remainder of the argument we focus on this subset of joints and give some properties holding for it, in terms of interaction between different formations with respect to channels. Since we need a full sequence of extended formations attached to these joints, k has to be at least eight times the number of channels inside a sequence of extended formations, that is, $k \geq 8 \cdot 48x = 2^7 \cdot 3x$.

First, we give some further definitions.

Nested formations A formation F is *nested* in a formation F' if there exist four path-edges $e_1, e_2 \in F$ and $e'_1, e'_2 \in F'$ cutting a boundary cb of a channel c such that all

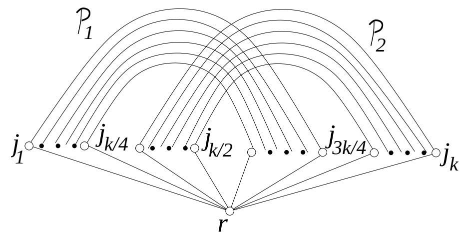


Figure 7: Two sets of passages $\mathcal{P}_1$ and $\mathcal{P}_2$ as described in Lemma 5.

the vertices of the path in F between e_1 and e_2 lie inside the region delimited by cb and by the path in F' between e'_1 and e'_2 (see Fig. 8(a)).

A series of pairwise nested formations $F_1, \ldots, F_k$ is *r-nested* if there exist r formations $F_{q_1}, ..., F_{q_r}$, with $1 \leq q_1, \ldots, q_r \leq k$, such that the 4-tuples of $F_{q_1}, ..., F_{q_r}$ have at least one common joint j, and such that for each pair $F_{q_p}, F_{q_{p+1}}$ there exists at least one formation F_z, with $1 \leq z \leq k$, such that the 4-tuple of F_z does not contain j, F_{q_p} is nested in F_z and F_z is nested in $F_{q_{p+1}}$ (see Fig. 8(b)).

Independent sets of formations Let $S_1, \ldots, S_k$ be sets of formations of one extended formation $EF(H)$ such that each set S_i, for $i = 1, \ldots, k$, contains formations $F_i(H_1), \ldots, F_i(H_r)$, with $(H_1, \ldots, H_r) \subset H$. Let $F_a(H_c)$ be not nested in $F_b(H_d)$, for each $1 \leq a, b \leq k$, $a \neq b$, and $1 \leq c, d \leq r$. If for each two sets S_a, S_b there exists a line l_1 (a line l_2) separating the vertices of S_a (of S_b) inside channel segment cs_1 (channel segment cs_2), then sets $S_1, \ldots, S_k$ are *independent* (see Fig. 8(c)).

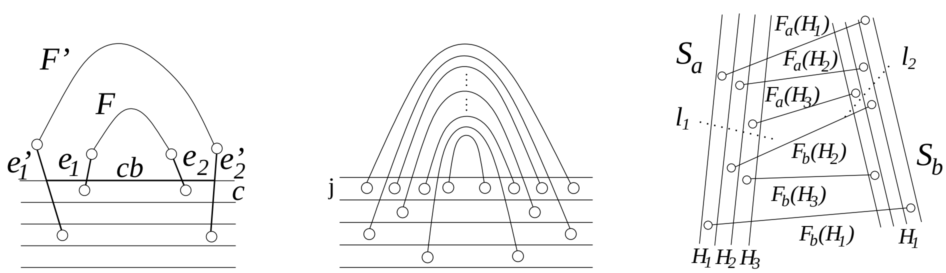


Figure 8: (a) A formation F nested in a formation F'. (b) A series of r-nested formations. (c) Two independent sets S_a and S_b.

In the following lemmata we prove that in any extended formation there exists a nesting of a certain depth (Lemma 8). This important property will be the starting point for the final argument and will be deeply exploited in the rest of the paper. We get to this conclusion by first proving that in an extended formation the number of independent sets of formations is limited (Lemma 6) and then by showing that, although there exist formations that are neither nested nor independent, in any extended formation there exists a certain number of pairs of formation that have to be either independent or nested (Lemma 7).

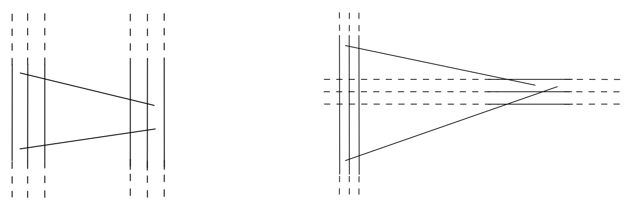
Lemma 6 *No extended formation contains $n \geq 2^{22} \cdot 14$ independent sets of formations $S_1, \ldots, S_n$ such that each set S_i contains formations $F_i(H_1), \ldots, F_i(H_r)$, where $r \geq 22$.*

Lemma 7 *Let EF be an extended formation and let $Q_1, \ldots, Q_4$ be four subsequences of formations, each consisting of a whole repetition $(H_1, H_2, \ldots, H_x)$ of EF. There exists either a pair of nested subsequences or a pair of independent subsequences among such formations.*

Lemma 8 *For every extended formation EF, there exists a k-nesting, with $k \geq 6$, among the formations of EF.*

Once the existence of 2-channels and of a nesting of a certain depth in each extended formation have been shown, we turn our attention to study how such a deep nesting can be performed inside the channels. In our discussion, we will get to the conclusion that, in any possible shape of the tree, either it is not possible to draw the formations creating the nesting without crossings, or that any planar drawing of such formations induces further geometrical constraints that do not allow for a planar drawing of the rest of the tree.

We start by giving some more formal definitions about the shapes of the channels. Let cs_a and cs_b, with $1 \leq a, b \leq 4$, be two channel segments. If the elongation of cs_a intersects cs_b, then it is possible to connect from cs_b to cs_a by cutting both the sides of cs_a. In this case, cs_a and cs_b have a *2-side connection* (see Fig. 9(b)). On the contrary, if the elongation of cs_a does not intersect cs_b, only one side of cs_a can be used. In this case, cs_a and cs_b have a *1-side connection* (see Fig. 9(a)).



(a) (b)

Figure 9: (a) A 1−side connection. (b) A 2−side connection.

First, we consider the case in which only 1-side connections are possible (an example of this case is provided by the M-shape, depicted in Fig. 6(b)).

Proposition 1 *If every two channel segments have a 1−side connection, then $\mathcal{T}$ and $\mathcal{P}$ do not admit any geometric simultaneous embedding.*

We prove this proposition by showing that, in this configuration, the existence of a deep nesting in a single extended formation, proved in Lemma 8, results in a crossing in either $\mathcal{T}$ or $\mathcal{P}$.

Lemma 9 *If an extended formation lies in a part of the channel that contains only 1-side connections, then $\mathcal{T}$ and $\mathcal{P}$ do not admit any geometric simultaneous embedding.*

Next, we study the case in which there exist 2-side connections. We distinguish two types of 2-side connections, based on whether the elongation of channel segment cs_a intersecting channel segment cs_b starts at the bendpoint that is closer to the root, or not. In the first case we have a *low Intersection* $I^l_{(a,b)}$ (see Fig. 10(a)), while in the second case

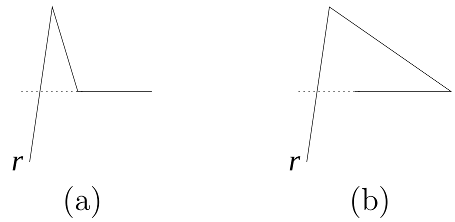


Figure 10: (a) A low Intersection $I^l_{(3,1)}$. (d) A high Intersection $I^h_{(3,1)}$.

we have a *high Intersection* $I^h_{(a,b)}$ (see Fig. 10(b)). We use notation $I_{(a,b)}$ to describe both $I^h_{(a,b)}$ and $I^l_{(a,b)}$. We say that two intersections $I_{(a,b)}$ and $I_{(c,d)}$ are *disjoint* if $a, d \in \{1, 2\}$ and $b, c \in \{3, 4\}$. For example, $I_{(1,3)}$ and $I_{(4,2)}$ are disjoint, while $I_{(1,3)}$ and $I_{(2,4)}$ are not.

Since consecutive channel segments can not create 2-side connections, in order to explore all the possible shapes we consider all the combinations of low and high intersections created by channel segments cs_1 and cs_2 with channel segments cs_3 and cs_4.

With the intent of proving that intersections of different channels have to maintain certain consistencies, we state the following lemma.

Lemma 10 *Consider two channels ch_p, ch_q with the same intersections. Then, none of channels ch_i, where $p < i < q$, have an intersection that is disjoint with the intersections of ch_p and of ch_q.*

As for Proposition 1, in order to prove that 2-side connections are not sufficient to obtain a simultaneous embedding of $\mathcal{T}$ and $\mathcal{P}$, we exploit the existence of the deep nesting shown in Lemma 8.

Observe that every extended formation that uses a channel segment cs_a to place the nesting has to place vertices inside the adjacent bending area. In the following lemma we prove that not many of the formations involved in the nesting can use the part of the path that creates the nesting to place vertices in such a bending area, and hence they have to reach the bending area in a different way.

Lemma 11 *Consider an x-nesting of formations inside a sequence of extended formations on an intersection $I_{(a,b)}$, with $a \leq 2$. Then, one of the nesting formations contains a pair of path-edges (u, v), (v, w), with v lying inside channel segment cs_a, that separates some of the formations in cs_a from the bending area $b(a, a+1)$ or $b(a-1, a)$ (see Fig. 11).*

Let the *inner area* and *outer area* of cs_a be the two parts in which cs_a is split by edges (u, v), (v, w), as described in Lemma 11. Assume that (u, v) and (v, w) do not cut any channel segment cs_b completely, since such a cut would create more restrictions than placing u or w inside cs_b.

Since in every extended formation EF there exists a path connecting the inner and the outer area by going around either vertex u or vertex w, we can infer that the extended formations using such paths create a structure that is analogous to the one created by the nested formations. Namely, because of the presence of a defect in every repetition of an extended formation, if only 1-side connections are available to host the vertices of such paths, then a crossing in $\mathcal{T}$ or $\mathcal{P}$ is created.

Lemma 12 *Let cs_a be a channel segment that is split into its inner area and outer area by two edges, as described in Lemma 11, such that every extended formation EF of a sequence of extended formations has vertices in both the areas. If the only possibility to connect vertices from the inner to the outer area is with a 1-side connection, then $\mathcal{T}$ and $\mathcal{P}$ do not admit any geometric simultaneous embedding.*

From Lemma 12 we conclude that having one single 2-side connection is not sufficient to obtain a geometric simultaneous embedding of the tree and the path. In the following we prove that a further 2-side connection is not useful if it is not disjoint from the first one.

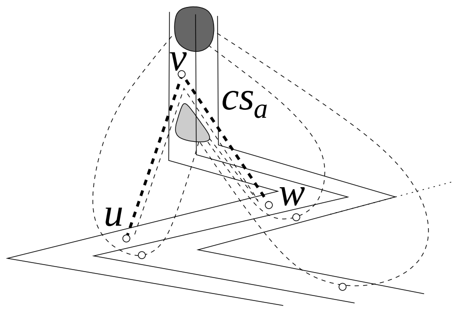


Figure 11: A situation as in Lemma 11. Inner and the outer areas are represented by a light grey and a dark grey region, respectively.

Proposition 2 *If there exists no pair of disjoint 2-side connections, then $\mathcal{T}$ and $\mathcal{P}$ do not admit any geometric simultaneous embedding.*

Observe that, in this setting, it is sufficient to restrict the analysis to cases $I_{(1,3)}$ (see Figs. 12(a)–(b)) and $I_{(3,1)}$ (see Figs. 12(c)–(d)), since the cases involving 2 and 4 can be reduced to them.

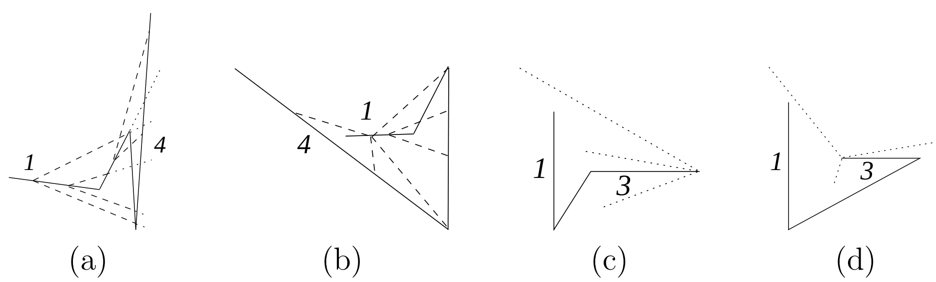


Figure 12: (a) Case $I_{(1,3)}$ $I^h_{(2,4)}$. Since a nesting at $I_{(1,3)}$ has to reach bending area $b(2,3)$, it crosses any nesting at $I^h_{(2,4)}$. (b) Case $I_{(1,3)}$ $I^l_{(2,4)}$. A nesting at $I_{(1,3)}$ crosses any nesting at $I^l_{(2,4)}$. Also, if there exist extended formations nesting at $I_{(1,4)}$, then they create a nesting also at $I_{(1,3)}$, as they have to reach $b(2,3)$ and $b(3,4)$. (c) and (d) If cs_4 is not on the convex hull, then either cs_1 or cs_2 is on the convex hull. Hence, a nesting at $I_{(3,1)}$ can not be drawn together with any other nesting, as either they cross or the second nesting takes place at a 1-side connection. (c) Case $I^l_{(3,1)}$. (d) Case $I^h_{(3,1)}$.

Lemma 13 *If a shape contains an intersection $I_{(1,3)}$ and does not contain any other intersection that is disjoint with $I_{(1,3)}$, then $\mathcal{T}$ and $\mathcal{P}$ do not admit any geometric simultaneous embedding.*

Lemma 14 *If there exists a sequence of extended formations in any shape containing an intersection $I_{(3,1)}$, then $\mathcal{T}$ and $\mathcal{P}$ do not admit any geometric simultaneous embedding.*

Observe that, in the latter lemma, we proved a property that is stronger than the one stated in Proposition 2. In fact, we proved that a simultaneous embedding cannot be obtained in any shape containing an intersection $I_{(3,1)}$, even if a second intersection that is disjoint with $I_{(3,1)}$ is present.

Finally, we tackle the general case where two disjoint intersections exist.

Proposition 3 *If there exists two disjoint 2-side connections, then $\mathcal{T}$ and $\mathcal{P}$ do not admit any geometric simultaneous embedding.*

Since the cases involving intersection $I_{(3,1)}$ were considered in Lemma 14, we only have to consider the eight different configurations where one intersection is $I_{(1,3)}$ and the other is one of $I^{h,l}_{(4,\{1,2\})}$. In the next three lemmata we cover the cases involving $I^{h}_{(1,3)}$ and in Lemma 18 the ones involving $I^{l}_{(1,3)}$.

Consider two consecutive channel segments cs_i and cs_{i+1} of a channel c and let e be a path-edge crossing the boundary of one of cs_i and cs_{i+1}, say cs_i. We say that e creates a *double cut* at c if the line through e cuts c in cs_{i+1}. A double cut is *simple* if the elongation of e cuts cs_{i+1} (see Fig. 13(a)) and *non-simple* if e itself cuts cs_{i+1} (see Fig. 13(b)). Also, a double cut of an extended formation EF is *extremal* with respect to a bending area $b(x, x+1)$ if there exists no double cut of EF that is closer than it to $b(x, x+1)$. We can state for double cuts a property that is analogous to the one stated for blocking cuts.

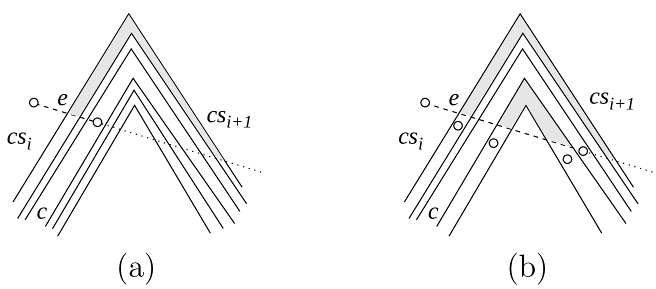


Figure 13: (a) A simple double cut. (b) A non-simple double cut.

Property 5 *Any edge e_k creating a double cut at a channel k in channel segment cs_i blocks visibility to the bending area $b(i, i+1)$ for a part of cs_i in each channel ch_h with $h > k$ (with $h < k$).*

In the following lemma we show that a particular ordering of extremal double cuts in two consecutive channel segments leads to a non-planarity in $\mathcal{T}$ or $\mathcal{P}$. Note that an ordering of extremal double cuts corresponds to an ordering of the connections of a subset of extended formations to the bending area. Then, we will show that both shapes $I^{h}_{(1,3)}$ $I^{h,l}_{(4,2)}$ induce this order (Lemma 17).

Lemma 15 *Let cs_i and cs_{i+1} be two consecutive channel segments. If there exists an ordered set $S := (1, 2, \ldots, 5)^3$ of extremal double cuts cutting cs_i and cs_{i+1} such that the order of the intersections of the double cuts with cs_i (with cs_{i+1}) is coherent with the order of S, then $\mathcal{T}$ and $\mathcal{P}$ do not admit any geometric simultaneous embedding.*

First, we state the existence of double cuts in these shapes. While the existence of double cuts in shape $I^h_{(1,3)}\ I^l_{(4,2)}$ can be easily seen (Fig 14(a)), in order to prove it in shape $I^h_{(1,3)}\ I^h_{(4,2)}$ we state the following lemma (see Fig 14(b)).

Lemma 16 *Each extended formation in shape $I^h_{(1,3)}\ I^h_{(4,2)}$ creates double cuts in at least one bending area.*

Then, we show that the existence of a double defect in every repetition of an extended formation leads to the existence of the undesired ordering of extremal double cuts in shape $I^h_{(1,3)}\ I^{h,l}_{(4,2)}$.

Lemma 17 *Every sequence of extending formations in shape $I^h_{(1,3)}\ I^{h,l}_{(4,2)}$ contains an ordered set $(1, 2, \ldots, 5)^3$ of extremal double cuts with respect to bending area either $b(2, 3)$ or $b(3, 4)$.*

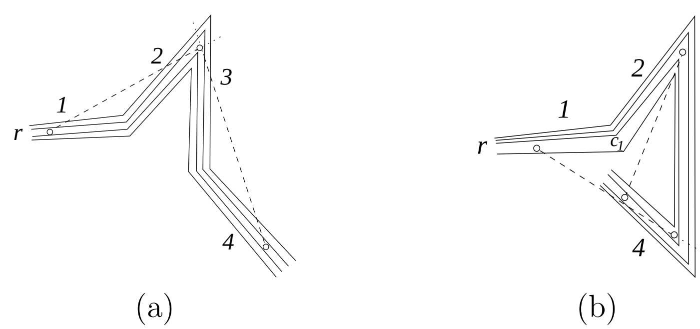


Figure 14: (a) Shape $I^h_{(1,3)}\ I^l_{(4,2)}$ creates double cuts at $b(2, 3)$. (b) Shape $I^h_{(1,3)}\ I^h_{(4,\{1,2\})}$ creates double cuts.

Finally, we consider the configurations where one intersection is $I^l_{(1,3)}$ and the other is one of $I^{h,l}_{(4,2)}$. We solve this cases by exploiting a geometrical property they exhibit. Namely, we observe that, in both configurations, channel segment cs_2 is on the convex hull of the shape created by the channel, and we show that, when this geometric property holds, a crossing either in $\mathcal{T}$ or in $\mathcal{P}$ is always found.

Lemma 18 *If channel segment cs_2 is part of the convex hull, then $\mathcal{T}$ and $\mathcal{P}$ do not admit any geometric simultaneous embedding.*

Based on the above discussion, we state the following theorem.

Theorem 1 *There exist a tree and a path that do not admit any geometric simultaneous embedding.*

Proof: Let $\mathcal{T}$ and $\mathcal{P}$ be the tree and the path described in Sect. 3. Then, by Lemma 5, Lemma 10, and Property 1, a part of $\mathcal{T}$ has to be drawn inside channels having at most four channel segments. Also, by Lemma 8, there exists a nesting of depth at least 6 inside each extended formation.

By Proposition 1, if there exist only 1-side connections, then $\mathcal{T}$ and $\mathcal{P}$ do not admit any simultaneous embedding. By Proposition 2, if there exists no pair of disjoint intersections, then $\mathcal{T}$ and $\mathcal{P}$ do not admit any simultaneous embedding. By Proposition 3, even if there exist two disjoint intersections, then $\mathcal{T}$ and $\mathcal{P}$ do not admit any simultaneous embedding. Since it is not possible to have more than two disjoint intersections, the statement follows.

□

5 Detailed Proofs

In this section we give the details of the proofs of some of the lemmas and properties stated in Sect. 4.

5.1 Proofs of Lemmata 2 and 3

Lemma 2. *For each formation $F(H)$, with $H = (h_1, \ldots, h_4)$, there exists a passage between some cells $c_1(h_a), c_2(h_a), c'(h_b) \in F(H)$, with $1 \leq a, b \leq 4$.*

Proof: First observe that, by Property 2, there exists no set of four cells connected to the same joint inside $F(H)$ that can be separated by a straight line. Hence, the cells of $F(H)$ connected to the same joint, say j_{h_a}, can be grouped into at most 3 different sets $S^1_{h_a}$, $S^2_{h_a}$, and $S^3_{h_a}$ such that cells from different sets can be separated by straight lines, but cells from the same set can not. As any two cells $c_1(h_a), c_2(h_a) \in F(H)$ can only be separated either by a straight-line or by a cell $c_3(h_a)$ of the same joint j_{h_a}, each two cells inside one of these sets can only be separated by other cells of the same set.

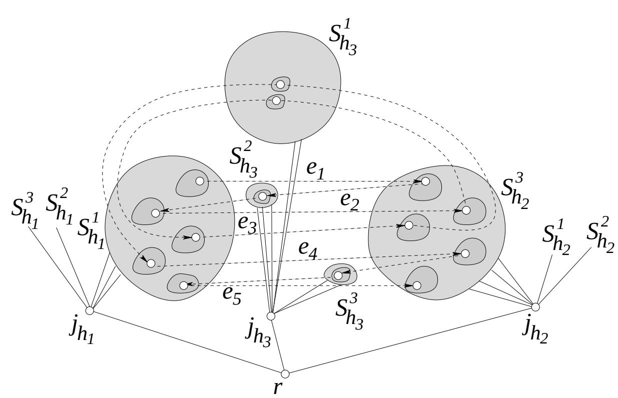


Figure 15: The five path-edges $e_1, \ldots, e_5$ connecting five cells of set $S^a_{h_1}$ with five cells of set $S^b_{h_2}$.

Consider the connections of the path through $F(H)$ with regard to this notion of sets of cells. Observe that, for any two joints $j_{h_x}, j_{h_{x+1}}$, with $1 \leq x \leq 4$, there are nine possible ways to connect between a set $S^y_{h_x}$, with $1 \leq y \leq 3$, and a set $S^{y'}_{h_{x+1}}$, with $1 \leq y' \leq 3$.

Then, since the part of $\mathcal{P}$ through $F(H)$ visits 37 times cells from $j_{h_1}, j_{h_2}, j_{h_3}$, in this order, there exist at least two sets $S^y_{h_1}$ and $S^{y'}_{h_2}$, with $1 \le y, y' \le 3$, that are connected by at least five path-edges $e_1, \ldots, e_5$ (see Fig.15). Observe that edges $e_1, \ldots, e_5$, together with the cells of $S^y_{h_1}$ and of $S^{y'}_{h_2}$, subdivide the plane into five connected regions. Since the path is continuous in $F(H)$, it connects from the end of e_1 (a cell of joint j_{h_2}) to the beginning of e_2 (a cell of joint j_{h_1}), from the end of e_2 to the beginning of e_3, and so on. If in the region between two edges e_s and e_{s+1}, with $1 \le s \le 4$, there exists no cell of joint j_{h_3}, then the path through $F(H)$ will not traverse such a region in the opposite direction, since $\mathcal{P}$ contains no edges going from a cell of j_{h_2} to a cell of j_{h_1}. Since there exist five edges between $S^y_{h_1}$ and $S^{y'}_{h_2}$ but at most 3 sets of cells on joint j_{h_3}, there exist at least two empty regions, which implies that the part of the path connecting e_s and e_{s+1} in a certain repetition of the formation creates a spiral, in the sense that it separates the cells connected to joint j_{h_4} in the previous repetitions from the analogous cells in the following repetitions.

Since four repetitions create four of such separated cells on j_{h_4}, by Property 2 at least two of such cells are not separated by a straight line, but are separated by the path. Hence, since the path of the spiral separating them can only consist only of a cell belonging to joint j_{h_3}, a passage inside $F(H)$ is created. □

Lemma 3. *Each passage contains at least one closed door.*

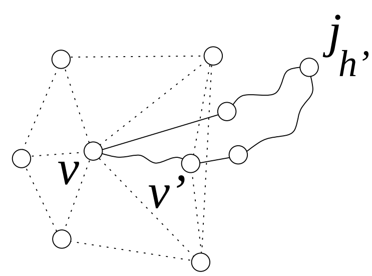


Figure 16: There exists a closed door in each passage.

Proof: Refer to Fig. 16. Let P_1 be a passage between cells $c_1(h)$, $c_2(h)$, and $c'(h')$. Consider any vertex v of c' inside the convex hull of $C := c_1 \cup c_2$. Further, consider all the triangles $\triangle(v, v_1, v_2)$ created by v with any two vertices $v_1, v_2 \in C$ such that $\triangle(v, v_1, v_2)$ does not enclose any other vertex of C. The path of tree-edges connecting v to $j_{h'}$ enters one of the triangles. Then, either it leaves the triangle on the opposite side, thereby creating a closed door, or it encounters a vertex v' of c'. Since at least one vertex of c' lies outside the convex hull of C, otherwise $c_1(h)$ and $c_2(h)$ would not be separated by $c'(h')$, it is possible to repeat the argument on triangle $\triangle(v', v_1, v_2)$ until a closed door is found.

□

5.2 Proof of Lemma 4

Lemma 4. *Given a set of joints $J = \{j_1, \ldots, j_y\}$, with $|J| = y := \binom{2^7 \cdot 3 \cdot x + 2}{3}$, there exists a subset $J' = \{j'_1, \ldots, j'_r\}$, with $|J'| = r \ge 2^7 \cdot 3 \cdot x$, such that for each pair of joints $j'_i, j'_h \in J'$ there exist two cells $c_1(i), c_2(i)$ creating a passage with a cell $c'(h)$.*

Proof: By construction of the tree, for each set of four joints, there are formations that visit only cells of these joints. By Lemma 2, there exists a passage inside each of these formations, which implies that for each set of four joints there exists a subset of two joints creating a passage. The number of joints needed to ensure the existence of a subset of joints of size r such that passages exist between each pair of joints is given by the Ramsey Number $R(r, 4)$. This number is defined as the minimal number of vertices of a graph G such that G either has a complete subgraph of size r or an independent set of size 4. Since in our case we can never have an independent set of size 4, we conclude that a subset of size r exists with the claimed property. The Ramsey number $R(r, 4)$ is not exactly known, but we can use the upper bound directly extracted from the proof of the Ramsey theorem [14] to obtain the stated bound. □

5.3 Proof of Lemma 5

In order to prove Lemma 5, we first need to prove the auxiliary property stating that the existence of a blocking cut implies the necessity of a further channel segment in some of the channels.

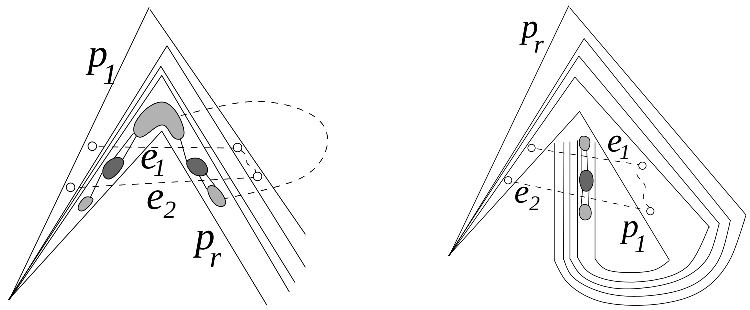
Property 4. *Let c be a channel that is cut twice by a blocking cut. If c has vertices in both the channel segments cut by the path-edge, then it has some vertices in a different channel segment.*

Proof: Consider the vertices lying in the two channel segments cs_a and cs_{a+1} of c. We have that either such vertices are in the bending area $b(a, a+1)$ of c or not. In the latter case, however, a vertex in $b(a, a+1)$ is needed in order to connect them in $\mathcal{T}$. Hence, at least one vertex v lies in $b(a, a+1)$. Also, at least one vertex w is separated from v in cs_a by the blocking cut. In fact, at least one vertex has to lie in the bending area $b(a-1, a)$ in order to connect v to the root in $\mathcal{T}$. Note that, if $a = 1$, w can be the root itself. Then, in order to have path connectivity between v and w, some vertices in a different channel segment are needed. □

Lemma 5. *Consider a set of joints $J = \{j_1, \ldots, j_k\}$ such that there exists a passage between each pair (j_i, j_h), with $1 \leq i, h \leq k$. Let $\mathcal{P}_1 = \{P \mid P$ connects c_i and $c_{\frac{3k}{4}+1-i}$, for $i = 1, \ldots, \frac{k}{4}\}$ and $\mathcal{P}_2 = \{P \mid P$ connects $c_{\frac{k}{4}+i}$ and c_{k+1-i}, for $i = 1, \ldots, \frac{k}{4}\}$ be two sets of passages between pairs of joints in J (see Fig. 7). Then, for at least $\frac{k}{4}$ of the joints of one set of passages, say $\mathcal{P}_1$, there exist paths in $\mathcal{T}$, starting at the root and containing these joints, which traverse all the doors of $\mathcal{P}_2$ with at least 2 and at most 3 bends. Also, at least half of these joints create an x-channel, with $2 \leq x \leq 3$.*

Proof: First observe that each passage of $\mathcal{P}_1$ is interconnected with each passage of $\mathcal{P}_2$ and that all the passages of $\mathcal{P}_1$ and all the passages of $\mathcal{P}_2$ are nested.

By Lemma 3 and Property 1, for one of $\mathcal{P}_1$ and $\mathcal{P}_2$, say $\mathcal{P}_1$, either for every joint of $\mathcal{P}_1$ between the joints of $\mathcal{P}_2$ in the order around the root or for every joint of $\mathcal{P}_1$ not between the joints of $\mathcal{P}_2$, there exists a path p_i in $\mathcal{T}$, starting at the root and containing these joints, which has to traverse all the doors of $\mathcal{P}_2$ by making at least 1 and at most 3 bends. Also, paths $p_1, \ldots, p_{\frac{k}{4}}$ can be ordered in such a way that a bendpoint of p_i encloses a bendpoint of p_h for each $h > i$. It follows that there exist x-channels with $1 \leq x \leq 3$ for each joint. Consider now the set of joints $J' \subset J$ visited by these paths. We assume the joints of $J' = \{j'_1, \ldots j'_r\}$ to be in this order around the root.



(a) (b)

Figure 17: (a) The separating cell c' is in the outermost channel. (b) The separating cell c' is in the innermost channel.

Consider the path p_1 whose bendpoint encloses the bendpoint of each of all the other paths and the path p_r whose bendpoint encloses the bendpoint of none of the other paths (see Figs. 17(a) and 17(b)). Please note that either p_1 visits j'_1 and p_r visits j'_r or vice versa, say p_1 visits j'_1. By construction, there exists a passage between cells from j'_1 and cells from j'_r. In this passage there exist either two path-edges e_1, e_2 of a cell $c'(1)$ separating two cells $c_1(r), c_2(r)$, thereby crossing the channel of j'_r, or two edges of a cell $c'(r)$ separating two cells $c_1(1), c_2(1)$, thereby crossing the channel of j'_1. We show that 1-channels are not sufficient to draw these passages.

In the first case (see Fig. 17(a)), both separating edges e_1, e_2 cross the path p_r before and after the bend, thereby creating blocking cuts separating vertices of the same cell, say c_1. By Property 4, an additional bend is needed in the channel. In the other case (see Fig. 17(b)), no edge connecting vertices of $c'(j'_r)$ crosses edges of p_1, and therefore at least another bend is needed in the channel. Hence, at least one of the channels needs an additional bend. Since there are passages between each pair of joints in J', all but one joint j_q have a path that has to bend an additional time. We note that the additional bendpoint of each path p_k aside from p_1, p_r, and p_q has to enclose all the additional bendpoints either of $p_1, \ldots, p_{k-1}$ or of $p_{k+1}, \ldots, p_r$. It follows that, for at least half of the joints, there exist x-channels where $2 \leq x \leq 3$. □

5.4 Proofs of Lemmata 6, 7, and 8

Lemma 6. *No extended formation contains $n \geq 2^{22} \cdot 14$ independent sets of formations $S_1, \ldots, S_n$ such that each set S_i contains formations $F_i(H_1), \ldots, F_i(H_r)$, where $r \geq 22$.*

Proof: Suppose that such independent sets $S_1, \ldots, S_n$ exist. We show that this induces a crossing in either $\mathcal{T}$ or $\mathcal{P}$. By Lemma 2, each formation contains a passage, and thereby an edge cutting the boundary of a channel. By Property 3, every edge has an end-vertex either in channel segment cs_1 or in cs_2. Hence, for each 4-tuple t, the formations placed in t in at least $n/2$ sets of formations have a common connection, that is, they have connections to vertices in the same channel segment, either cs_1 or cs_2. Let $S^1 = \{S_p, \ldots, S_q\}$, with $1 \leq p < q \leq n$ and $q-p \geq n/2$, be the set of set of formations containing such independent sets.

By using the same argument we can find a subset $S^2 \subset S^1$ of size $\frac{n}{4}$ such that, for each pair of 4-tuples t, t', the sets belonging to S^2 have at least two common connections. By continuing this procedure we arrive at a subset S^r containing at least $\frac{n}{2^r}$ sets having at least r common connections. Since all these common connections have to connect to either cs_1 or cs_2, we have identified a set $S = \{S'_1, \ldots, S'_{\frac{n}{2^r}}\}$ of size $\frac{n}{2^r}$ of sets of formations of size at least $\frac{r}{2}$ that has all its connections to the same channel segment CS.

We now consider, for each of the formations of S, the edges cutting the boundary of CS. Since any of those edges can intersect the channel boundary on two different sides, for every formation $F_{\frac{r}{4}}$ in each set S'_i, at least half of such edges cut the same side of the channel, thereby crossing either all the channels $1, \ldots, \frac{r}{4}-1$ or all the channels $\frac{r}{4}+1, \ldots, \frac{r}{2}$, say the former.

Consider now the formations $F_{\frac{r}{8}}$ in each of the sets. These formations in the sets $S'_2, S'_4, \ldots, S'_{\frac{n}{2^{r+1}}}$ are separated in CS by the edges of the formations $F_{\frac{r}{4}}$ of the sets $S'_3, S'_5, \ldots, S'_{\frac{n}{2^r}-1}$. To avoid a monotonic ordering of the separated formations and thereby the existence of a region-level nonplanar tree, formations $F_{\frac{r}{8}}$ have to place vertices in an adjacent channel segment CS'. However, in this way they create blocking cuts for either all the channels $1, \ldots, \frac{r}{8}-1$ or all the channels $\frac{r}{8}+1, \ldots, \frac{r}{4}$, say the former.

Consider now the formations F_1 in each of the sets. These formations in the sets $S'_3, S'_5, \ldots, S'_{\frac{n}{2^r}-2}$ are separated in CS by the edges of the formations $F_{\frac{r}{8}}$ of the sets $S'_4, S'_6, \ldots, S'_{\frac{n}{2^r}-3}$. By the same argument as above, also these formations have to place vertices in an adjacent channel segment that is visible from some of the separated areas of CS. Since the connections of the formations $F_{\frac{r}{8}}$ block visibility for the connections to CS', the formations F_1 have to use the other adjacent channel segment CS'', thereby blocking all the channels $1, \ldots r_2$.

Finally, consider the formations F_2 in the sets $S'_4, S'_6, \ldots, S'_{10}$. These formations are now separated in CS by the edges connecting the formations $F_{\frac{r}{8}}$ to CS' and by the edges connecting the formations F_1 to CS''. Therefore, these formations cannot use any channel segment other than CS. So, by Property 2, there exists a region-level nonplanar tree. □

Lemma 7. *Let EF be an extended formation and let $Q_1, \ldots, Q_4$ be four subsequences of formations, each consisting of a whole repetition $(H_1, H_2, \ldots, H_x)$ of EF. There exists either a pair of nested subsequences or a pair of independent subsequences among such formations.*

Proof: Assume that no pair of nested subsequences exists. We show that a pair of independent subsequences exists.

First, we consider how $Q_1, \ldots, Q_4$ use the first two channel segments cs_1 and cs_2 to place their formations. Each of these subsequences uses either only cs_1, only cs_2, or both. Observe that, if a subsequence uses only cs_1 and another one uses only cs_2, then such subsequences are clearly independent. So, we can assume that all of $Q_1, \ldots, Q_4$ use a common channel segment, say cs_2.

Then, we show that if there exist three subsequences that use only cs_2, then at least two of them are separated in cs_2. In fact, consider two subsequences using cs_2 that are not independent. Since such subsequences are not nested, by assumption, they contain formations on the same channel a that intersect with different boundaries of a. However, a third subsequence containing a formation that intersects a boundary of a is such that there exists either a nesting or a clear separation between this subsequence and the other

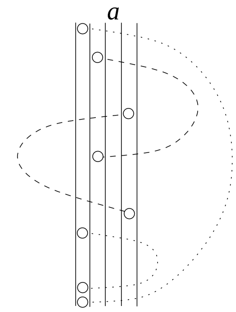


Figure 18: If three subsequences use the same channel segment cs, then at least two of them are either nesting or separated in cs.

subsequence intersecting the same boundary (see Fig. 18).

From this and from the fact that there are four subsequences using cs_2, we derive that two subsequences, say Q_1, Q_2, are separated in cs_2 and are not separated in cs_1. Then, the third subsequence Q_3 can be placed in such a way that it is not separated from Q_1 and Q_2 in cs_2. However, this implies that Q_4 is separated in cs_1 from two of Q_1, Q_2, Q_3 and in cs_2 from two of Q_1, Q_2, Q_3, which implies that Q_4 is separated in both channel segments from one of Q_1, Q_2, Q_3. □

Lemma 8. *For every extended formation EF, there exists a k-nesting, with $k \geq 6$, among the formations of EF.*

Proof: Suppose that there is no k-nesting among the formations in EF. We show that there exist more than n sets of independent formations in EF from the same set of channels C, where $n \geq 2^{22} \cdot 14$ and $|C| \geq 22$. By Lemma 6, this fact clearly implies the statement.

Observe that, by Lemma 7, there exist at most $(n-1) \cdot 3$ different nestings of repetitions $(H_1, H_2, \ldots, H_x)$ of formations in EF such that there are less than n independent sets of subsequences. Also note that, if some formations belonging to two different repetitions are nesting, then all the formations of these repetitions have to be part of some nesting. However, observe that this does not necessarily mean for all the formations to nest with each other and to build a single nesting.

Since the number of channels used inside EF is greater than $(n-1) \cdot 3 \cdot 3$, where $n \geq 2^{22} \cdot 14$, we have a nesting consisting of repetitions of formations in EF with at least 3 different defects.

Let the nesting consist of repetitions $Q_1^1, \ldots, Q_1^r, Q_2^1, \ldots, Q_2^r, \ldots, Q_k^1 \ldots, Q_k^r$, where Q_i^h denotes the h-th occurrence of a repetition of EF with a defect at the 4-tuple H_i. Further, let the path connect such repetitions in the order $Q_1^1, Q_2^1, \ldots, Q_k^1, Q_1^2, \ldots, Q_k^2, \ldots, Q_k^r$. We show that there exists a pair of independent formations within this nesting.

Consider now the first two nesting repetitions, that is, Q_1^1 and Q_2^1. Let the nesting consist of a formation $F(H_k)$ from Q_1^1 nesting in a formation $F'(H_s)$ from Q_2^1, where $1 \leq k, s \leq x$. Consider the edges $e_1, e_2 \in F(H_k)$ and $e'_1, e'_2 \in F'(H_s)$ determining the nesting. Assume, without loss of generality, that the path p between e'_2 and e_2 does not contain e'_1 and e_1. Consider the two parts a and b of the channel boundary that is cut by all such edges, where a is between e_1 and e'_1 and b is between e_2 and e'_2. Consider now the closed region delimited by the path through $F'(H_s)$, path p, the path through $F(H_k)$, and a. Such a region is split into two closed regions R_{in} and R_{nest} by b (see Fig. 19).

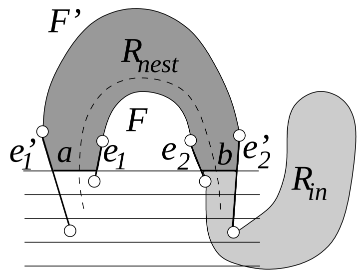


Figure 19: Regions R_{in} and R_{nest}.

Observe that, in order to go from R_{in} to the outer region, any path has to cross both a and b by using a vertex inside R_{nest}. We note that the part of $\mathcal{P}$ starting at e_1' and not containing $F(H_k)$ is either completely contained in the outer region or has to cross over between R_{in} and the outer region by traversing R_{nest}. Similarly, the part of $\mathcal{P}$ starting at e_1 and not containing $F'(H_s)$ either does not reach the outer region or has to cross over between R_{in} and the outer region by traversing R_{nest}. Furthermore, any formation F'' using such a path either crosses over, thereby cutting both a and b, or it does not enter R_{in} at all. Observe that, in the first case, F is nested in F'' and F'' is nested in F'.

Consider now the third nesting repetition Q_3^1 of sequence $(H_1, H_2, \ldots, H_x)$ (see Figs. 20(a) and 20(b)). It is easy to see that, if Q_3^1 is nested between Q_1^1 and Q_2^1, then there exists a nesting of depth 1, as Q_3^1 contains a defect at a different 4-tuple. Hence, we have only to consider the cases in which the remaining repetitions create the nesting by creating a spiral, that is, by strictly going either outward or inward. By this we mean that the i-th repetition Q_i^1 has to be placed such that either Q_i^1 is nested inside Q_{i-1}^1 (inward) or vice versa (outward). Without loss of generality, we assume the latter (see Fig. 20(c)).

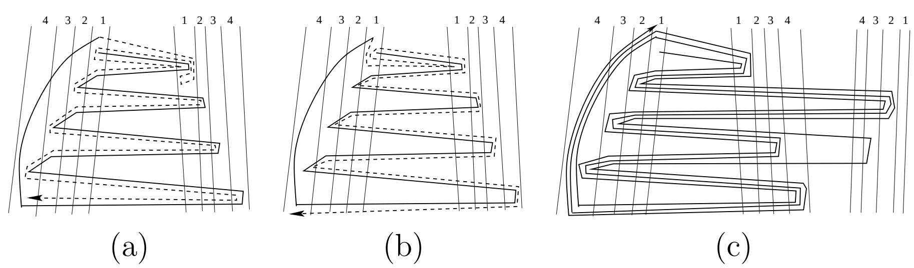
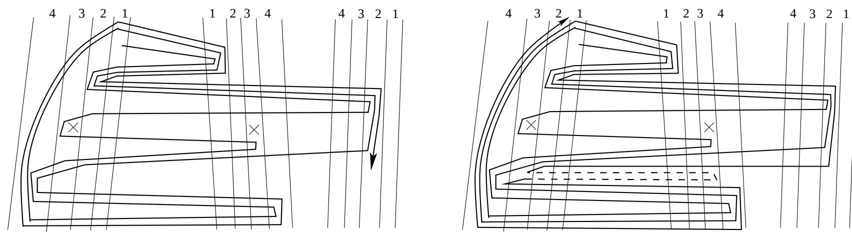


Figure 20: (a) and (b) Possible configurations for Q_1^1, Q_2^1, and Q_3^1. (c) The repetitions follow the outward orientation.

Consider now a defect in a 4-tuple H_c, with $1 < c < k$, at a certain repetition Q_i^h. Since the path is moving outward, the connection between H_{c-1} and H_{c+1} blocks visibility for the following repetitions to the part of the channel segment where vertices of H_c were placed till that repetition (see Fig. 21(a) for an example with $c = 3$).

A possible placement for the vertices of H_c in the following repetitions that does not increase the depth of the nesting could be in the same part of the channel segment where vertices of a 4-tuple $H_{c'}$, with $c' \neq c$, were placed till that repetition. We call *shift* such a move. However, in order to place vertices of H_c and of $H_{c'}$ in the same zone, all the vertices of H_c belonging to the current cell have to be placed there (see dashed lines in



(a) (b)

Figure 21: The connection between channels 2 and 4 blocks visibility for the following repetitions to the part of the channel segment where vertices of channel 3 were placed till that repetition.

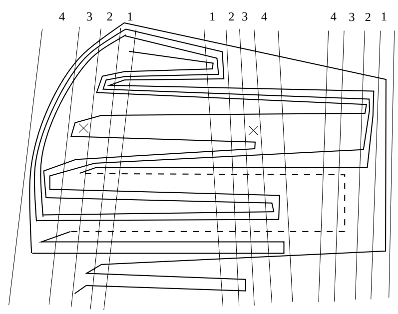


Figure 22: All the channels $c, \ldots, x$ are shifted and the next repetition starts in a completely different region.

Fig. 21(b), where $c' = c + 1$), which implies that a further defect at H_c in one of the following repetitions encloses all the vertices of each of the previously drawn cells, hence separating them with a straight line from the following cells. Hence, also the vertices of $H_{c'}$ have to perform a shift to a 4-tuple $H_{c''}$, with $c \neq c'' \neq c'$. Again, if the vertices of $H_{c'}$ and of $H_{c''}$ lie in the same zone, we have two cells that are separated by a straight line, and hence also the vertices of $H_{c''}$ have to perform a shift. By repeating such an argument we conclude that the only possibility for not having vertices of different 4-tuples lying in the same zone is to shift all the 4-tuples $H_c, \ldots, H_x$ and to go back to H_1 for starting the following repetition in a completely different region (see Fig. 22, where the following repetition is performed completely below the previous one). However, this implies that there exist two repetitions in one configuration that have to be separated by a straight line and therefore are independent, in contradiction to our assumption. Hence, after $3 \cdot x + 1$ repetitions, we arrive at a nesting of depth 1. By repeating this argument, after $3 \cdot x \cdot 6$ repetitions we obtain a nesting of depth 6. □

5.5 Proof of Proposition 1

Lemma 9. *If an extended formation lies in a part of the channel that contains only 1-side connections, then $\mathcal{T}$ and $\mathcal{P}$ do not admit any geometric simultaneous embedding.*

Proof: First observe that, by Lemma 8, there exists a k-nesting with $k \geq 6$ in any extended formation EF.

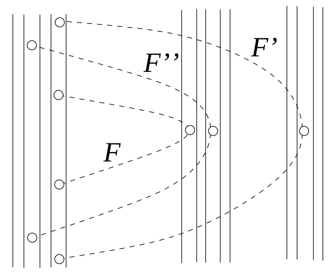


Figure 23: Illustration for the case in which only 1-side connections are possible.

Consider two nested formations $F, F' \in EF$ belonging to the k-nesting and the formation $F'' \in EF$ not sharing any joint with F and F' and such that F is nested in F'' and F'' is nested in F'. Since each pair of channel segments have a 1-side connection, F'' blocks visibility for F' on the channel segment used by F for the nesting (see Fig. 23). Hence, F' has to use a different channel segment to perform its nesting, which increases by one the number of used channel segments for each level of nesting. Since the tree supports at most 4 channel segments, the statement follows. □

5.6 Proof of Lemma 10

Lemma 10. *Consider two channels ch_p, ch_q with the same intersections. Then, none of channels ch_i, where $p < i < q$, have an intersection that is disjoint with the intersections of ch_p and of ch_q.*

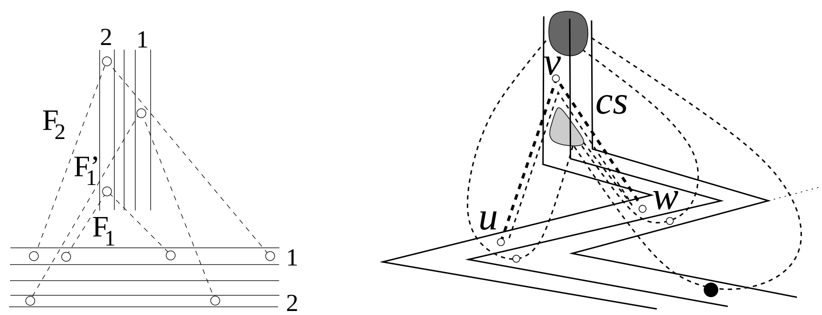
Proof: The statement follows from the fact that the channel boundaries of ch_p and ch_q delimit the channel for all the joints between p and q. So, if any channel ch_i, with $p < i < q$, had an intersection different from the one of ch_p and ch_q, either it would intersect with one of the channel boundaries of ch_p or ch_q or it would have to bend around one of the channel boundaries, hence crossing twice a straight line. □

5.7 Proof of Lemma 11

Lemma 11. *Consider an x-nesting of formations inside a sequence of extended formations on an intersection $I_{(a,b)}$, with $a \leq 2$. Then, one of the nesting formations contains a pair of path-edges (u, v), (v, w), with v lying inside channel segment cs_a, that separates some of the formations in cs_a from the bending area $b(a, a+1)$ or $b(a-1, a)$ (see Fig. 11).*

Proof: Consider three extended formations $EF_1(H_1), EF_2(H_1), EF_3(H_1)$ lying in a 4-tuple of channels H_1 and two extended formations $EF_1(H_2), EF_2(H_2)$ lying in a 4-tuple of channels H_2 such that all the channels of the sequence of extended formations are between H_1 and H_2 and there is no formation $F \notin EF(H_1), EF(H_2)$ nesting between $EF_1(H_1), EF_2(H_1), EF_3(H_1)$ and $EF_1(H_2), EF_2(H_2)$. Suppose, without loss of generality, that the bending point of H_1 is enclosed into the bending point of H_2.

Refer to Fig. 24(a). Consider a formation $F_1 \in EF_1(H_1)$ nesting with a formation $F'_1 \in EF_1(H_2)$. The connections from F'_1 to channel segment cs_a and back has to go around the vertex placed by F_1 on channel segment cs_a. Therefore, at least one of the connections of F_1' cuts all the channels between H_1 and H_2, that is, all the channels where the sequence of extended formations is placed. Such a connection separates the vertices of F_1 from the vertices of a formation $F_2 \in EF_2(H_1)$ in cs_a. Therefore, at least one of the connections of F_2 to cs_a cuts either all the channels in cs_a or all the channels in cs_{a+1} (or cs_{a-1}), hence becoming a blocking cut for such channels. It follows that all the formations nesting inside F_2 on such channels can not place vertices in the bending area $b(a, a+1)$ (or $b(a-1, a)$) outside F_2. □



(a) (b)

Figure 24: (a) The connections from F_1 to cs_a enforce F_1' to cut all the channels and the connections from F_1' to cs_a enforce F_2 to cut all the channels. (b) A pair of path-edges (u, v), (v, w), with v lying inside channel segment cs_a, separates some of the formations in cs_a from the bending area $b(a, a+1)$ or $b(a-1, a)$. The chosen turning vertex is represented by a big black circle and is in configuration β. The inner and the outer areas are represented by a light grey and a dark grey region, respectively.

5.8 Proof of Lemma 12

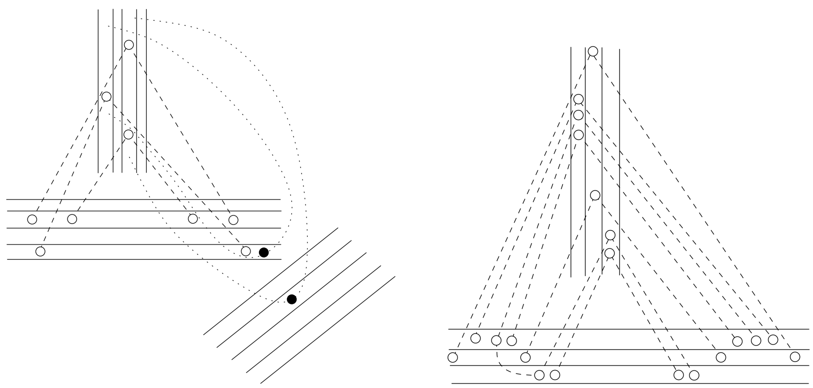
In order to prove Lemma 12, we first need to introduce some definition and to prove an auxiliary lemma.

Refer to Fig. 24(b). Let (u, v) and (v, w) be a pair of edges separating a channel segment cs_a into its inner and outer area. We call such a pair the *open triangle* of the extended formation. For each path of an extended formation connecting vertices in the inner area to vertices in the outer area, consider a vertex, called *turning vertex*, which is placed in cs_b and for which no other path in EF exists that connects the inner and the outer area by using a channel segment cs_c such that the subpath to cs_c intersects either cs_c or its elongation. If there exist more than one of such vertices, then arbitrarily choose one of them. Observe that the path connecting from the inner area to the outer area through the turning vertex encloses exactly one of u and w. If it encloses u, it is in configuration α, otherwise it is in configuration β. If there exist both paths in α and paths in β configuration, then we arbitrarily consider one of them. Finally, consider the connections between different extended formations inside a sequence of extended formations. Consider a turning vertex v in a channel segment cs_a of a channel ch such that the edges incident to v cut a channel ch_b. Then, any connection of an extended formation of ch_b from the inner to the outer area in the same configuration as ch and with its turning vertex v' in cs_a is such that v' lies inside the convex hull of the open triangle.

Lemma 19 *If the connection between the inner and the outer area can be realized only through a* 1*-side connection, then not all the extended formations in a sequence of extended formations can place turning vertices in the same channel segment.*

Proof: Assume, for a contradiction, that all the turning vertices are in the same channel segment. Consider a sequence of extended formations $SEF(H)$, where $H = (H_1^*, \ldots, H_{12}^*)$, and the extended formations $EF_j(H_i^*) \in SEF$, with $j = 1, \ldots, 110$.

We first show that in SEF there exist some extended formations $EF_j(H_i^*) \in SEF$ using connections in α configuration and some using connections in β configuration. Consider the continuous subsequence of extended formations $EF_1(H_1^*), \ldots, EF_1(H_3^*)$ in SEF. Assume that all the turning vertices of these extended formations are in α configuration. Consider a further subsequence $EF_p(H_1^*), \ldots, EF_p(H_3^*)$, with $1 \leq p < 110$, of SEF with a defect at H_2^*. Then, the connection between $EF_p(H_1^*)$ and $EF_p(H_3^*)$ crosses the channels in H_2^*, thereby blocking any extended formation $EF_q(H_2^*)$, with $p < q \leq 110$, from being in α configuration. Hence, when considering another subsequence of SEF on the same set of channels which does not contain defects at $H_1^*, \ldots, H_3^*$, either the extended formation placed in H_2^* is in β configuration or it uses another channel segment to place the turning vertex.



(a) (b)

Figure 25: (a) Two triangles from the same channel have to use different channel segments if a triangle of another channel is between them. Turning vertices are represented by black circles. (b) When a defect at H_2 in encountered, the connection between $EF(H_1)$ and $EF(H_3)$ does not permit the following $EF(H_2)$ to respect the ordering of triangles.

Consider the open triangle of each extended formation. Note that, if an open triangle of an extended formation $EF(H_k)$ belongs to the inner area of an extended formation $EF(H_s)$ and the open triangle of $EF(H_s)$ belongs to the inner area of an extended formation $EF'(H_k)$, with $k < s$, then $EF(H_k)$ has to use a different channel segment to place its turning vertex (see Fig. 25(a)). Hence, the open triangles have to be ordered according to the order of the sets of channels used by the extended formations. Also, if the continuous path connecting two open triangles $t_1 = (u, v, w)$ and $t_2 = (u', v', w')$ of consecutive extended formations $EF(H_s), EF(H_{s+1})$ connects vertex u to vertex w' (or u' to w) via the outer area, then an open triangle of $EF(H_1)$ that occurs before $EF(H_s)$ and an open triangle of $EF'(H_1)$ that occurs after $EF(H_{s+1})$ are nested with the open triangle given by the connection of t_1 and t_2 in an ordering different from the order of the channels.

We show that it is not possible for the open triangles of all the extended formation to be nested in such an order because of the presence of defects in the repetitions of the sequence of extended formations. Namely, consider two x-tuples of 4-tuples of indices of channels H_1^*, H_2^* such that there exists an extended formation $EF_p(H_1^*)$ in α configuration and an extended formation $EF_p(H_2^*)$ in β configuration, and consider two further extended formations $EF_p(H_3^*)$ and $EF_p(H_4^*)$ directly following $EF_p(H_1^*)$ and $EF_p(H_2^*)$ in SEF. Also, consider the first extended formation $EF_{p+1}(H_1^*)$ is on the set of channels H_1^* following $EF_p(H_4^*)$ in SEF. Consider now the first repetition q of SEF after repetition $p+1$ having a defect at H_2^*. As $EF_p(H_1^*)$ is in α configuration and $EF_p(H_2^*)$ is in β configuration, the connection of $EF_q(H_1^*)$ to $EF_q(H_3^*)$ in this repetition blocks access for the all the extended formations $EF_m(H_2^*)$, with $m > q$, to the area where it would have to place vertices in order to respect the ordering of triangles (see Fig. 25(b)). Therefore, after 3 full repetitions of the sequence in SEF, at least one extended formation has to use a different channel segment to place its turning vertex. □

Now we are ready to prove the claimed lemma.

Lemma 12. *Let cs_a be a channel segment that is split into its inner area and outer area by two edges, as described in Lemma 11, such that every extended formation EF of a sequence of extended formations has vertices in both the areas. If the only possibility to connect vertices from the inner to the outer area is with a 1-side connection, then $\mathcal{T}$ and $\mathcal{P}$ do not admit any geometric simultaneous embedding.*

Proof: Consider two extended formations $EF_p(H_{12}^*)$ and $EF_{p+1}(H_1^*)$ that are consecutive in SEF. First note that the connection between $EF_p(H_{12}^*)$ and $EF_{p+1}(H_1^*)$ cuts all the between H_2^* and H_{11}^* in either channel segment cs_1 or cs_2, say cs_1. Since both of these extended formations are also connected to the bending area $b(3,4)$ between cs_3 and cs_4, it is not possible for an extended formation $EF_m(H_s^*)$, with $m > p+1$ and $s \in \{2, \ldots, 11\}$, to connect from vertices above the connection between $EF_p(H_{12}^*)$ and $EF_{p+1}(H_1^*)$ to vertices below it by using a path that passes through $b(3,4)$. Further, note that if all such extended formations $EF_m(H_s^*)$ are in cs_2, then a connection is needed from cs_1 to cs_2 in the set of channels channel H_{12}^*. However, by Lemma 19, after three defects in the subsequence of $\{H_2^*, \ldots, H_{11}^*\}$ it is no longer possible for any extended formation $EF_m(H_s^*)$, with $s \in \{2, \ldots, 11\}$, to place its turning vertex in the same channel segment. Since the path is continuous and since the connection between an extended formation $EF_q(H_{12}^*)$ and $EF_q(H_1^*)$ is repeated at a certain repetition $q > p$, we can follow that the path creates a spiral that is directed either inward or outward. Also, in order to respect the order of the sequence, it will be impossible for the path to reverse the direction of the spiral. Hence, once a direction of the spiral has been chosen, either inward or outward, all the connections in the remaining part of the sequence have to follow the same. This implies that, if a connection between two consecutive extended formations $EF_m(H_s^*)$ and $EF_m(H_{s+1}^*)$ is performed in a different channel segment than the one between $EF_m(H_{s-1}^*)$ and $EF_m(H_s^*)$, then all the connections of this type have to change. However, when a defect at H_{s+1}^* is encountered, also the connection between $EF_m(H_s^*)$ and $EF_m(H_{s+2}^*)$ has to change channel segment, thereby making impossible for any future connection between $EF_m(H_s^*)$ and $EF_m(H_{s+1}^*)$ to change channel segment. We conclude that, after a full repetition of SEF, which contains defects at each set of channels, all the extended formations should place their turning vertices in the same channel segment, which is not possible, by Lemma 19, hence proving the statement. □

5.9 Proofs of Proposition 2

Lemma 13. *If a shape contains an intersection $I_{(1,3)}$ and does not contain any other intersection that is disjoint with $I_{(1,3)}$, then $\mathcal{T}$ and $\mathcal{P}$ do not admit any geometric simultaneous embedding.*

Proof: First observe that the only intersections that are not disjoint with $I_{(1,3)}$ and that could occur together with $I_{(1,3)}$ are $I_{(2,4)}$ and $I_{(1,4)}$. Consider the nesting of depth greater than or equal to 6 that is present in any extended formation (Lemma 8). Observe that a nesting can only take place at intersections $I_{(1,3)}$ and either $I_{(2,4)}$ or $I_{(1,4)}$. Remind that, by Property 3, every edge has an end-vertex in either cs_1 or cs_2. Also, by construction, the stabilizers are placed in cs_1 or cs_2. Note that the stabilizers also work as 1-vertices in the tails of other cells. This means that if there exist seven sets of tails that can be separated by straight lines, then there exist a region-level nonplanar tree, by Lemma 6. Observe that, by nesting them according to the sequence, the previous condition would be fulfilled.

This means that we have either a sorting or other nestings. We first show that there exist at most one x-nesting with $x \geq 6$.

Consider the case $I^h_{(2,4)}$ (see Fig 12(a)). Observe that intersections $I_{(1,4)}$ and $I_{(1,3)}$ are either both high or both low and use channel segment cs_1. Also, every connection from cs_1 to cs_4 cuts either cs_2 or cs_3 and, if one of these connections cuts cs_2, then every nesting cutting cs_1 closer to $b(1,2)$ than the previous connection, then it has to cut cs_2, as well. Hence, we can consider all the connections to cs_4 as connections to cs_2 or cs_3. Also, since any connection cutting a channel segment is more restrictive than a connection placing a vertex inside the same channel segment, the connections to cs_2 or cs_3 can be considered as the same. Finally, since each extended formation in the nesting has to connect to bending area $b_{(2,3)}$, it is not possible to have a nesting at $I^h_{(2,4)}$ together with a nesting at $I_{(1,3)}$. Hence, we conclude that only one nesting is possible in this case.

Consider the case $I^l_{(2,4)}$ (see Fig 12(b)). Observe that 1-vertices can be placed at most in cs_2 and 2-vertices can be placed at most in cs_3, and that every extended formations belonging to a nesting has to visit these vertices. Therefore, if there exist both a nesting at $I_{(1,3)}$ and a nesting at $I_{(1,4)}$, then the connections to the 1- and 2-vertices in the bending areas $b(2,3)$ and $b(3,4)$ are such that every extended formation nesting at $I_{(1,4)}$ makes a nesting with the extended formations nesting at $I_{(1,3)}$. Hence, also in this case only one nesting is possible.

So we consider the unique nesting of depth $x \leq 6$ and we show that any way of sorting the nesting formations in the channels will cause separated cells, hence proving the existence of a nonplanar region-level tree.

Consider four consecutive repetitions of the sequence of formations. These formations visit areas of cs_1 and are separated by previously placed formations from other formations on the same channels. This will result in some cells to become separated in cs_1. Since, by Property 2, the number of separated cells in cs_1 cannot be larger than 3, for any set of four such separated formations there exists a pair of formations F_1, F_2 that change their order in cs_1 by using one of the sides of the nesting. If between this pair of formations there is a formation of a different channel, then this formation has to choose the other side to reorder with a formation outside F_1, F_2. We further note that, if there are two such connections F_1, F_4 and F_2, F_3 on the same side that are connecting formations of one channel, nested in the order F_1, F_2, F_3, F_4, and another connection on the same side between F'_1, F'_2 such that F'_1 is nested between F_1, F_2 and F'_2 between F_3, F_4, then this creates a 1-nesting. In the following we show that a nesting of depth at least 6 is reached.

Assume the repetitions of formations in the extended formation to be placed in the order a, b, c, d, e. If this order is not coherent with the order in which the channels appear in the sequence of formations inside the extended formation, then we have already some connections that close both sides of the nesting for some formations. So we assume them to be in the order given by the sequence. Then, consider a repetition of formations with a defect at some 4-tuple H_i. Then, there exists a connection closing off at one side all the previously placed formations of H_i. However, there are sequences with defects also at 4-tuples H_{i+1} and H_{i-1}, which can not be realized on the same side as the defects at H_i. We generalize this by saying that all the defects at odd indices are in one side, while the defects at even indices are in the other side. Since the path is continuous and has to reach from the last formation in a repetition to the first one in the following repetition, the continuation of the path can only use either the odd or the even defects. This implies that, when considering three further repetitions of formations, the first and the third having a defect at H_i and the second having no defect at H_i, there exists a nesting of depth 1 between them. Since, by Lemma 9, there cannot be a nesting of depth greater than 5 at this place, we conclude that after six repetitions of such a triple of formations there will be at least two formations that are separated from each other. By repeating this argument we arrive, after $7 \cdot 6 \cdot 2$ repetitions, either at the existence of 7 formations that are separated on cs_1 and cs_2 or at the existence of a nesting of depth 6, both of which will not be drawable without the aid of another intersection that is able to support the second nesting. □

Lemma 14. *If there exists a sequence of extended formations in any shape containing an intersection $I_{(3,1)}$, then $\mathcal{T}$ and $\mathcal{P}$ do not admit any geometric simultaneous embedding.*

Proof: Consider a sequence of extended formations in a shape containing an intersection $I_{(3,1)}$. We show that $\mathcal{T}$ and $\mathcal{P}$ do not admit any geometric simultaneous embedding. Observe that there exist several possibilities for channel segment cs_4 to be placed. Either there exists no intersection of an elongation of one channel segment with another channel segment or there exists at least one of the intersections $I_{(1,4)}$, $I_{(4,2)}$, $I_{(4,1)}$ or $I_{(2,4)}$.

We note that, if there exists the intersection $I_{(3,1)}$, then at least one of cs_1, cs_2, and cs_4 are part of the convex hull (see Figs. 12(c) and (d)).

First, we show that there exists a nesting at $I_{(3,1)}$ (case $I_{(4,1)}$ can be considered as the same).

Consider case $I^h_{(3,1)}$. We have that cs_1 and cs_2 are on the convex hull restricted to the first three channel segments, and cs_4 can force at most one of them out of it. Hence, one of cs_1 and cs_2 is part of the convex hull. We distinguish the two cases.

Suppose that cs_2 is part of the convex hull. Assume there exists a nesting at $I_{(1,4)}$. From cs_4 the only possible connection without a 1-side connection is the one to cs_2, which, however, is on the convex hull. Hence, an argument analogous to the one used in Lemma 13 proves that the nesting at $I_{(2,4)}$ has size smaller than $7 * 12$, which implies that the rest of the nesting has to take place at $I_{(3,1)}$.

Suppose that cs_1 is part of the convex hull. Assume that there exists a nesting at $I_{(2,4)}$. Every connection from cs_4 has to be either to cs_1 or to cs_2, by Property 3. However, a nesting is already taking place at cs_2, and hence we have connections to cs_1. As cs_1 is on the convex hull, an argument analogous to the one used in Lemma 13 proves that the nesting at $I_{(2,4)}$ has size smaller than $7 * 12$, which implies the rest of the nesting has to take place at $I_{(3,1)}$.

Consider case $I^l_{(3,1)}$. Since cs_2 is not part of the convex hull, either cs_1 or cs_4 are. If cs_1 is on the convex hull, then the same argument as before holds, while if cs_4 is on the convex hull, then no reordering is possible.

Hence, we conclude that the nesting of every extended formation has to take place at $I_{(3,1)}$.

Consider a sequence of extended formations SEF which uses only channels in this particular shape. We have that all the extended formations in SEF have to do a nesting at $I_{(\{3,4\},1)}$ by placing an open triangle with the middle vertex in either cs_3 or cs_4, hence splitting it into an inner and an outer area. Observe that, by Lemma 11, only a limited part of the nesting can be performed in the bending area. Also, every extended formation EF having at least one vertex either in cs_3 or in cs_4 has a vertex in the bending area. Hence, every extended has to use both of such areas. If cs_1 is on the convex hull, then there exist only 1-sided connections to connect such areas, which implies the statement, by Lemma 12. On the other hand, if cs_1 is not on the convex hull, then there exists intersection $I_{(1,4)}$, and cs_4 can be also used to perform connections from the inner to the outer area. However, since cs_4 is on the convex hull, such connections are only 1-side. Hence, by Lemma 12, the statement follows. □

5.10 Proofs of Proposition 3

Lemma 15. *Let cs_i and cs_{i+1} be two consecutive channel segments. If there exists an ordered set $S := (1, 2, \ldots, 5)^3$ of extremal double cuts cutting cs_i and cs_{i+1} such that the order of the intersections of the double cuts with cs_i (with cs_{i+1}) is coherent with the order of S, then $\mathcal{T}$ and $\mathcal{P}$ do not admit any geometric simultaneous embedding.*

Proof: Assume first that cs_i and cs_{i+1} are such that the bendpoint of channel 5 encloses the bendpoint of all the other channels. Hence, any edge creating a double cut at a channel c has to cut all the channels c' with $c' > c$, either in cs_i or in cs_{i+1}. Refer to Fig. 26.

Consider the first repetition $(1, 2, \ldots, 5)$. Let e_1 be an edge creating a double cut at channel 1. Assume, without loss of generality, that e_1 cuts channel segment cs_i. Observe that, for channel 1, the visibility constraints determined in channels $2, \ldots, 5$ in cs_i and in cs_{i+1} by the double cut created by e_1 do not depend on whether it is simple or non-simple. Indeed, by Property 5, edge e_1 blocks visibility to $b(i, i+1)$ for the part of cs_i where edges creating double cuts at channels $2, \ldots, 5$ following e_1 in S have to place their end-vertices.

Then, consider an edge e_3 creating a double cut at channel 3 in the first repetition of $(1, 2, \ldots, 5)$.

If e_3 cuts cs_i (see Fig. 26(a)), then it has to create either a non-simple double cut or a simple one. However, in the latter case, an edge e'_3 between cs_i and cs_{i+1} in channel 3, which creates a blocking cut in channel 2, is needed. Hence, in both cases, channel 2 is cut both in cs_i and in cs_{i+1}, either by e_3 or by e'_3. It follows that an edge e_2 creating a double

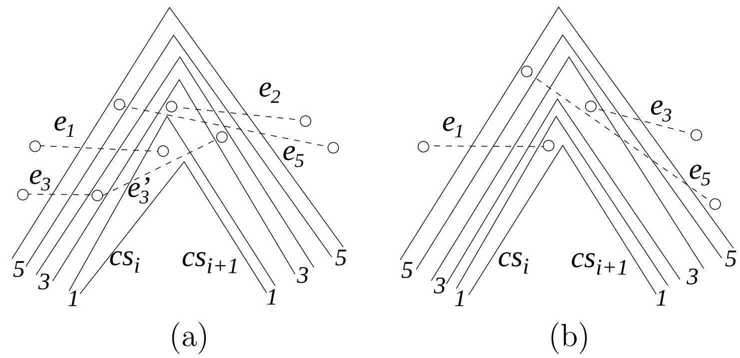


Figure 26: Proof of Lemma 15. (a) e_3 cuts cs_i. (b) e_3 cuts cs_{i+1}.

cut at channel 2 in the second repetition of $(1, 2, \ldots, 5)$ has to cut cs_{i+1}, hence blocking visibility to $b(i, i+1)$ for the part of cs_{i+1} where edges creating double cuts at channels $3, \ldots, 5$ following it in S have to place their end-vertices, by Property 5. Further, consider an edge e_5 creating a double cut at channel 5 in the second repetition of $(1, 2, \ldots, 5)$. Since visibility to $b(i, i+1)$ is blocked by e_1 and e_3 in cs_i and by e_2 in cs_{i+1}, e_2 has to create a non-simple double cut (or a simple one plus a blocking cut), hence cutting channel 4 both in cs_i and in cs_{i+1}. It follows that, by Property 4, an edge e_4 creating a double cut at channel 4 in the third repetition of $(1, 2, \ldots, 5)$ can place its end-vertex neither in cs_i nor in cs_{i+1}.

If e_3 cuts cs_{i+1} (see Fig. 26(b)), then it has to create a simple double cut. Again, by Property 5, edge e_3 blocks visibility to $b(i, i+1)$ for the part of cs_{i+1} where edges creating double cuts following e_3 in S have to place their end-vertices. Hence, an edge e_5 creating a double cut at channel 5 in the first repetition of $(1, 2, \ldots, 5)$ cannot create a simple double cut, since its visibility to $b(i, i+1)$ is blocked by e_1 in cs_i and by e_3 in cs_{i+1}. This implies that e_5 creates a non-simple double cut (or a simple one plus a blocking cut) at channel 5, cutting either cs_i or cs_{i+1}, hence cutting channel 4 both in cs_i and in cs_{i+1}. It follows that, by Property 4, an edge e_4 creating a double cut at channel 4 in the second repetition of $(1, 2, \ldots, 5)$ can place its end-vertex neither in cs_i nor in cs_{i+1}.

The case in which cs_i and cs_{i+1} are such that the bendpoint of 1 encloses the bendpoint of all the other channels can be proved analogously. Namely, the same argument holds with channel 5 playing the role of channel 1, channel 1 playing the role of channel 5, channel 3 having the same role as before, channel 4 playing the role of channel 2, and channel 2 playing the role of channel 4. Observe that, in order to obtain the needed ordering in this setting, 3 repetitions of $(1, 2, \ldots, 5)$ are needed. In fact, we consider channel 5 in the first repetition, channels 3 and 4 in the second one, and channels 1 and 2 in the third one. □

Lemma 16. *Each extended formation in shape $I^h_{(1,3)}$ $I^h_{(4,2)}$ creates double cuts in at least one bending area.*

Proof: Refer to Fig. 29(a). Assume, without loss of generality, that the first bendpoint of channel 1 encloses the first bendpoint of all the other channels. This implies that the second and the third bendpoints of channel 1 are enclosed by the second and the third bendpoints of all the other channels, respectively.

Suppose, for a contradiction, that there exists no double cut in $b(2, 3)$ and in $b(3, 4)$.

Hence, any edge e connecting to $b(2,3)$ (to $b(3,4)$) is such that e and its elongation cut each channel once. Consider an edge connecting to $b(2,3)$ in a channel i. Such an edge creates a triangle together with channel segments cs_3 and cs_4 of channel i which encloses the bending areas $b(3,4)$ of all the channels $h < i$ by cutting such channels twice. Hence, a connection to such a bending area in one of these channels has to be performed from outside the triangle. However, since in shape $I^h_{(1,3)}\ I^h_{(4,2)}$ both the bending areas $b(2,3)$ and $b(3,4)$ are on the convex hull, this is only possible with a double cut, which contradicts the hypothesis. □

Lemma 17. *Every sequence of extending formations in shape* $I^h_{(1,3)}\ I^{h,l}_{(4,2)}$ *contains an ordered set* $(1,2,\dots,5)^3$ *of extremal double cuts with respect to bending area either* $b(2,3)$ *or* $b(3,4)$.

Proof: Shape $I^h_{(1,3)}\ I^h_{(4,2)}$ is similar to shape $I^h_{(1,3)}\ I^h_{(4,1)}$, with the only difference on the slope of channel segment 4, which is such that its elongation crosses cs_2 and not cs_1. Shape $I^h_{(1,3)}\ I^l_{(4,2)}$ is depicted in Fig. 29(b).

Assume, without loss of generality, that the first bendpoint of channel 1 is enclosed by the first bendpoint of all the other channels. This implies that the second bendpoint of channel 1 encloses the second bendpoint of all the other channels.

First observe that bending area $b(2,3)$ is on the convex hull, both in shape $I^h_{(1,3)}\ I^h_{(4,2)}$ and in shape $I^h_{(1,3)}\ I^l_{(4,2)}$.

Also, observe that all the extended formations have some vertices in $b(2,3)$ and in $b(3,4)$, and hence all the extended formations have to reach such bending areas with path-edges.

In shape $I^h_{(1,3)}\ I^h_{(4,2)}$, by Lemma 16, there exist double cuts either in $b(2,3)$ or in $b(3,4)$, while in shape $I^h_{(1,3)}\ I^l_{(4,2)}$ there exist double cuts in $b(2,3)$, since the only possible connections to $b(2,3)$ are from channel segments cs_1 and cs_4, which both create double cuts (see Fig. 29(b)). Hence, we consider the extremal double cuts of each extended formation with respect to one of $b(2,3)$ or $b(3,4)$, say $b(2,3)$.

Consider two sets of extended formations creating double cuts in $b(2,3)$ at channels $1,\dots,5$, respectively. Observe that the extended formations in these two sets could be placed in such a way that the ordering of their extremal double cuts is $(1,1,2,2,\dots,5,5)$. The same holds for the following occurrences of extended formations creating double cuts in $b(2,3)$ at channels $1,\dots,5$, respectively. Clearly, in this way an ordering $(1^n,2^n,\dots,5^n)$ could be achieved and hence an ordered set $(1,2,\dots,5)^3$ of double cuts would be never obtained (see Fig. 27(a)).

However, every repetition of extended formations inside a sequence of extended formations contains a double defect at some channel. We show, with an argument similar to the one used in Lemma 8, that the presence of such double defects determines an ordering $(1,2,\dots,5)^3$ of extremal double cuts after a certain number of repetitions of extended formations inside a sequence of extended formations. Namely, consider a double defect at channel i in a certain repetition. The connection between channels $i-1$ and $i+2$ cannot be performed in the same area as the connection between channels $i-1$ and i and between channels i and $i+1$ was performed in the previous repetition. Hence, such a connection has to be performed either in the same area as the connection between channels $i+1$ and $i+2$ was performed (see Fig. 27(b)), or in cs_4 (this is only possible in shape $I^h_{(1,3)}\ I^l_{(4,2)}$, see Fig. 27(c)). Observe that, going to cs_4 to make the connection,

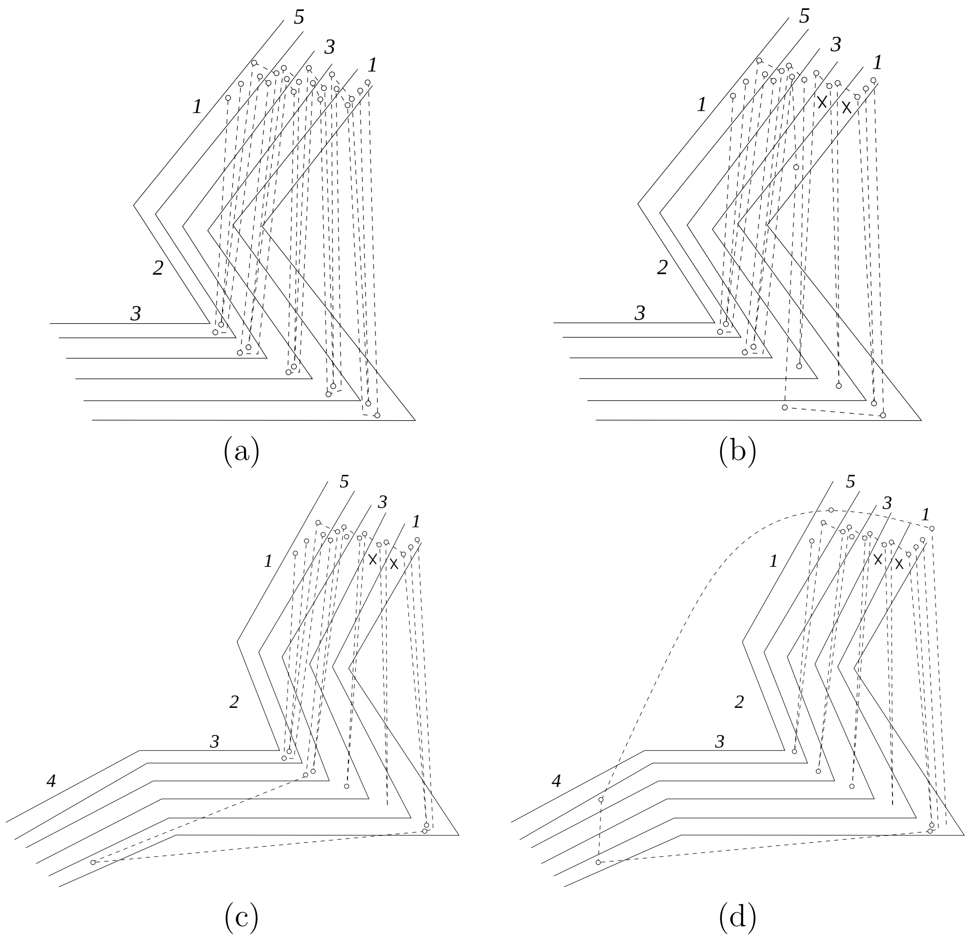


Figure 27: (a) The ordering of the extremal double cuts is $(1, 1, 2, 2, \ldots, 5, 5)$. (b) and (c) When a double defect is encountered, the connection between channels $i-1$ and $i+2$ cannot be performed in the same area as the connection between channels $i-1$ and i and between channels i and $i+1$ was performed in the previous repetition: (b) The connection is performed in the same area as the connection between channels $i+1$ and $i+2$ was performed. (c) The connection is performed in cs_4. (d) If cs_4 is used to spiral, the considered double cut was not extremal.

then to cs_1, and finally back to $b(2, 3)$, hence creating a spiral, implies that the considered double cut is not extremal (see Fig. 27(d)). Therefore, the only possibility to consider is to connect channels $i-1$ and $i+2$ in cs_4 and then to come back to $b(2, 3)$ with a double cut. Hence, independently on whether cs_4 is used or not, the connection between channels $i-1$ and $i+2$ blocks visibility for the following repetitions to the areas where the connections between some channels were performed in the previous repetition. This implies that the ordering $(1^n, 2^n, \ldots, 5^n)$ of extremal double cuts cannot be respected in the following repetitions. In fact, a partial order $(i, i+1, i+2)^2$ is obtained in a repetition of formations creating extremal double cuts at channels $1, \ldots, 5$.

Also, when two different double defects having a channel in common are considered, the effect of such defects is combined. Namely, consider a double defect at channel 3 in a certain repetition. Consider two sets of extended formations creating double cuts in $b(2, 3)$ at channels $1, \ldots, 5$, respectively. Observe that the extended formations in these

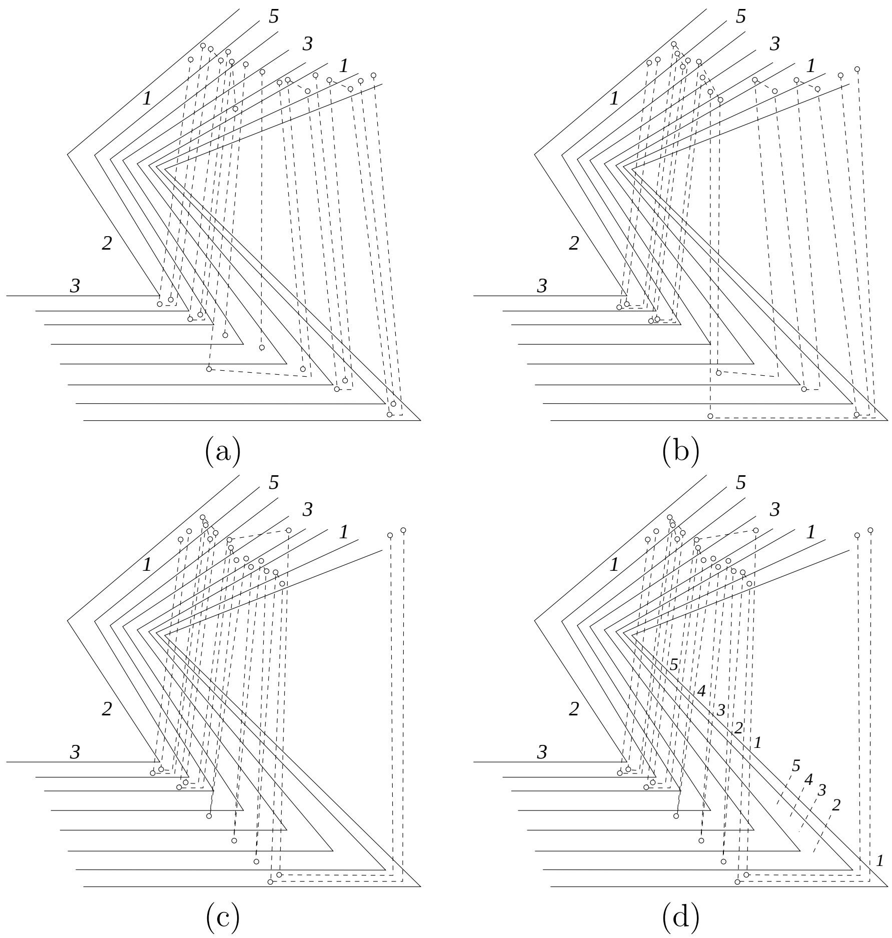


Figure 28: (a) A repetition with a double defect in channel 2 is considered. (b) A repetition with a double defect in channel 0 is considered. (c) A repetition without any double defect in channels $1, \ldots, 5$ is considered. (d) An ordered set $(1, \ldots, 5)$ is obtained.

two sets could be placed in such a way that the ordering of their extremal double cuts is $(1, 1, 2, 2, \ldots, 5, 5)$. The same holds for the following occurrences of extended formations creating double cuts in $b(2, 3)$ at channels $1, \ldots, 5$, respectively. Clearly, in this way an ordering $(1^n, 2^n, \ldots, 5^n)$ could be achieved and hence an ordered set $(1, 2, \ldots, 5)^3$ of double cuts would be never obtained (see Fig. 27(a)). The connection between channels 2 and 5 blocks visibility to the areas where the connection between 2 and 3 and between 3 and 4 were performed at the previous repetitions (see Fig. 28(a)). Then, consider a double defect at channel 1 in a following repetition. We have that the connection between channels 0 and 3 can not be performed where the connection between 2 and 3 was performed in the previous repetitions, since such an area is blocked by the presence of the connection between channels 2 and 5. Hence, a double cut at channel 3 has to be placed after the double cut at channel 5 created in the previous repetition (see Fig. 28(b)). Consider now a further repetition with a defect not involving any of channels $1, \ldots, 5$. The region where the connection from 1 to 2 was performed in the previous repetitions is blocked by the connection between 0 and 3 and hence a double cut at channel 1 has to be placed after the one at channel 3 of the previous repetition, which, in its turn, was created after the one at channel 5 (see Fig. 28(c)). Also, all the double cuts at channels $2, \ldots, 5$ have to be placed after the double cut at 1, and hence a shift of the whole sequence $1, \ldots, 5$ after the double cut at 5 is performed and an ordered set $(1, 2, \ldots, 5)^2$ is obtained (see Fig. 28(d)). Observe that at most two repetitions of extended formation inside a sequence of extended formations such that each set contains a double defect at each channel are needed to obtain such a shift. By repeating such an argument we obtain another shifting of the whole sequence $(1, \ldots, 5)$, which results in the desired ordered set $(1, 2, \ldots, 5)^3$. We have that a set of repetitions of extended formation containing a double defect at each channel is needed to obtain the first sequence $(1, 2, \ldots, 5)^2$, then two of such sets are needed to get to $(1, 2, \ldots, 5)^2$, and two more are needed to get to $(1, 2, \ldots, 5)^3$, which proves the statement.

Observe that, if it were possible to partition the defects into two sets such that there exists no pair of defects involving a common channel inside the same set, then such sets could be independently drawn inside two different areas and the effects of the defects could not be combined to obtain $(1, 2, \ldots, 5)^3$. However, since each double defect involves two consecutive channels, at least three sets are needed to obtain a partition with such a property. In that case, however, an ordered set $(1, 2, \ldots, 5)^3$ could be obtained by simply considering a repetition of $(1, 2, \ldots, 5)$ in each of the sets. □

Lemma 18. *If channel segment cs_2 is part of the convex hull, then $\mathcal{T}$ and $\mathcal{P}$ do not admit any geometric simultaneous embedding.*

Proof: First observe that, with an argument analogous to the one used in Lemma 13, it is possible to show that there exists a nesting at intersection $I_{(4,\{1,2\})}$. Then, by Property 3, every vertex that is placed in cs_4 is connected to two vertices that are placed either in cs_1 or in cs_2. Hence, the continuous path connecting to a vertex placed in cs_4 creates an open triangle, having one corner in cs_4 and two corners either in cs_1 or in its elongation, which cuts cs_4 into its inner and outer area.

By Lemma 11, not all of these triangles can be placed in the bending area $b(3, 4)$. Hence, every extended formation, starting from the second of the sequence, has to place their vertices in both the inner and the outer area of the triangle created by the first one.

Observe that, in order to connect the inner area to the outer area, the extended formations can only use 1-side connections. Namely, cs_1 creates a 1-side connection. Channel segment cs_2 is on the convex hull. Since, by Property 3, every vertex that is placed in cs_3 is connected to two vertices that are placed either in cs_1 or in cs_2, also cs_3 creates a 1-side connection. Hence, by Lemma 12, $\mathcal{T}$ and $\mathcal{P}$ do not admit any geometric simultaneous embedding. □

6 Geometric Simultaneous Embedding of a Tree of Height 2 and a Path

In this section we describe an algorithm for constructing a geometric simultaneous embedding of any tree $\mathcal{T}$ of height 2 and any path $\mathcal{P}$. Refer to Fig. 29.

Start by drawing the root r of $\mathcal{T}$ on the origin of a coordinate system. Choose a ray R_1 emanating from the origin and entering the first quadrant, and a ray R_2 emanating from the origin and entering the fourth quadrant. Consider the wedge W delimited by R_1 and R_2 and containing the positive x-axis. Split W into t wedges $W_1, \ldots, W_t$, in this clockwise order around the origin, where t is the number of vertices adjacent to r in $\mathcal{T}$, by emanating $t-1$ equispaced rays from the origin.

Then, consider the two subpaths $\mathcal{P}_1$ and $\mathcal{P}_2$ of $\mathcal{P}$ starting at r. Assign an orientation to $\mathcal{P}_1$ and $\mathcal{P}_2$ such that the two edges $(r,u) \in \mathcal{P}_1$ and $(r,v) \in \mathcal{P}_2$ incident to r in $\mathcal{P}$ are exiting r.

Finally, consider the t subtrees $\mathcal{T}_1, \dots, \mathcal{T}_t$ of $\mathcal{T}$ rooted at a node adjacent to r, such that $u \in \mathcal{T}_1$ and $v \in \mathcal{T}_t$.

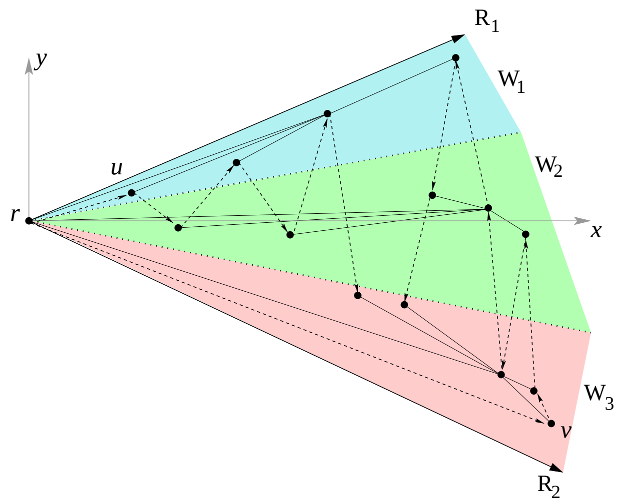


Figure 29: Construction of a geometric simultaneous embedding of a tree with height 2 and a path.

The vertices of each subtree $\mathcal{T}_i$ are drawn inside wedge W_i, in such a way that:

1. vertex u is the vertex with the lowest x-coordinate in the drawing, except for r;
2. vertices belonging to $\mathcal{P}_1$ are placed in increasing order of x-coordinate according to the orientation of $\mathcal{P}_1$;
3. vertex v is the vertex with the highest x-coordinate in the drawing;
4. vertices belonging to $\mathcal{P}_2 \setminus r$ are placed in decreasing order of x-coordinate according to the orientation of $\mathcal{P}_2$, in such a way that the leftmost vertex of $\mathcal{P}_2 \setminus r$ is to the right of the rightmost vertex of $\mathcal{P}_1$; and
5. no vertex is placed below segment $\overline{rv}$.

Since $\mathcal{T}$ has height 2, each subtree $\mathcal{T}_i$, with $i = 1 \dots, t$, is a star. Hence, it can be drawn inside its own wedge W_i without creating any intersection among tree-edges. Observe that the same holds also for subtree $\mathcal{T}_t$, where the wedge to consider is the part of W_t above segment $\overline{rv}$.

Since $\mathcal{P}_1$ and $\mathcal{P}_2 \setminus \{r\}$ are drawn in monotonic order of x-coordinate and are separated from each other, and edge (r,v) connecting such two paths is on the convex hull of the point-set, no intersection among path-edges is created.

From the discussion above, we have the following theorem.

Theorem 2 *A tree of height* 2 *and a path always admit a geometric simultaneous embedding.*

7 Conclusions

In this paper we have shown that there exist a tree $\mathcal{T}$ and a path $\mathcal{P}$ on the same set of vertices that do not admit any geometric simultaneous embedding, which means that there exists no set of points in the plane allowing a planar embedding of both $\mathcal{T}$ and $\mathcal{P}$. We obtained this result by extending the concept of level nonplanar trees [9] to the one of region-level nonplanar trees. Namely, we showed that there exist trees that do not admit any planar embedding if the vertices are forced to lie inside particularly defined regions according to a prescribed ordering. Then, we constructed $\mathcal{T}$ and $\mathcal{P}$ so that the path creates these particular regions and at least one of the many region-level nonplanar trees composing $\mathcal{T}$ has its vertices forced to lie inside them in the desired order. Observe that our result also implies that there exist two edge-disjoint trees that do not admit any geometric simultaneous embedding, which answers an open question posed in [13], where the case of two non-edge-disjoint trees was solved.

It is important to note that, even if our counterexample consists of a huge number of vertices, it can also be considered as "simple", in the sense that the height of the tree is just 4. In this direction, we proved that if the tree has height 2, then it admits a geometric simultaneous embedding with any path. This gives raise to an intriguing open question about whether a tree of height 3 and a path always admit a geometric simultaneous embedding or not.

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