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Collapsing cliques to get a planar graph is NP-complete

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ABSTRACT

A graph is a (planar, K_h)-graph if a collection of disjoint clusters can be identified such that the subgraph induced by each cluster is an h-clique and collapsing all clusters yields a planar graph. Recognizing (planar, K_h)-graphs is a special instance of the more general problem of recognizing $(\mathcal{X}, \mathcal{Y})$-graphs, where $\mathcal{X}$ and $\mathcal{Y}$ are two chosen families of graphs.

This model, together with the $(\mathcal{X}, \mathcal{Y})$-graph terminology, was introduced in [3] and generalized in [1, 2] to support hybrid visualization. In particular, if the clusters are requested to be a partition of the set of the vertices of the input graph G, as it is in [3], then G is called a strong $(\mathcal{X}, \mathcal{Y})$-graph, otherwise G is a weak $(\mathcal{X}, \mathcal{Y})$-graph or, simply, an $(\mathcal{X}, \mathcal{Y})$-graph.

In this paper we address (planar, K_h)-recognition in the weak model, and show that this problem is NP-complete if $h \geq 5$. This result parallels the analogous result for the strong model [10, 9]. We remark that allowing the contraction of any clique of size greater than or equal to $h \geq 5$ also yields an NP-complete problem.

1 Introduction

Given two graph classes $\mathcal{X}$ and $\mathcal{Y}$, a graph $G = (V, E)$ is an *$\mathcal{X}$-graph of $\mathcal{Y}$-graphs* (or *$(\mathcal{X}, \mathcal{Y})$-graph*, for short) if a family $V_1, V_2, \ldots, V_h$ of disjoint subsets of V, called *clusters*, can be identified, such that:

1. each cluster induces a graph belonging to class $\mathcal{Y}$, and
2. the *reduced graph* G^* obtained from G by collapsing each cluster into a single vertex and replacing multiple edges with a single one is a graph of class $\mathcal{X}$.

If subset $V_1, V_2, \ldots V_h$ are asked to be a partition of V, that is, if we add the constraint that $V = V_1 \cup V_2 \cup \cdots \cup V_h$, then G is called a *strong $(\mathcal{X}, \mathcal{Y})$-graph*, otherwise G is called a *weak $(\mathcal{X}, \mathcal{Y})$-graph* or, simply, an *$(\mathcal{X}, \mathcal{Y})$-graph*. The strong model of $\mathcal{X}$-graph of $\mathcal{Y}$-graphs, also known as *two level clustered graphs* [9, 16, 11], was introduced in [3].

The weak model, instead, was introduced in [1, 2] to support hybrid visualization. In fact, the visualization and the exploration of large graphs could take advantage of suitable clusterings of their vertices that allow us to convey part of the information (the high level structure) with the usual node-link metaphor and part of it (the subgraphs induced by the clusters) by some other means, such as adjacency matrices [7]. Since it is proved that drawing the graphs planarly is the most effective strategy when using the node-link metaphor [13, 14, 15, 17], and that adjacency matrices are the most effective representations for denser graphs [5, 6], it makes sense to pursue a clustering that guarantees both a planar high level graph and dense clusters.

In particular, identifying cliques in large graphs may lead to a very effective strategy for information visualization, since when the user is aware that the clusters are cliques, their internal edges are understood and do not need to be explicitly displayed. Unfortunately, we show that identifying clusters that are K_h subgraphs, with $h \geq 5$, and that once collapsed yield a planar graph is NP-complete. More formally we prove the following theorem.

Theorem 1 *Deciding whether a graph is a (planar,K_h)-graph, for $h \geq 5$, is NP-complete.*

In Sect. 2 we review known results about $(\mathcal{X}, \mathcal{Y})$-graph recognition in the strong and in the weak model. In Sect. 3 we prove that (planar, K_h)-graph recognition is in NP. Section 4 contains a reduction from the Planar 3-Satisfiability problem [12] to prove that it is NP-hard deciding whether a graph becomes planar by collapsing some K_h vertex-disjoint clusters when $h \geq 5$. Finally Sect. 5 contains our conclusions.

2 Related Works

Both for the strong model and for the weak one, by considering different families for $\mathcal{X}$- and $\mathcal{Y}$-graphs one obtains different $(\mathcal{X}, \mathcal{Y})$-decomposition problems which have diverse importance with respect to applications or to the insight into graph-theoretic decomposition problems.

Table 1: Known results for cliques $\mathcal{Y}$-graphs in the strong model.

$\mathcal{Y}$-graphs	$\mathcal{X}$-graphs			
	general graph	planar graph	3-cycle	cycle
cliques	NP-complete [8]	NP-hard [10, 9]	NP-hard [3]	Pol. if diameter > 3 [3]
3-cliques	NP-complete [4]			

Table 2: Known results for paths and cycles $\mathcal{Y}$-graphs in the strong model.

$\mathcal{Y}$-graphs	$\mathcal{X}$-graphs			
	tree	non-trivial path	single k-tree	single k-star
paths	NP-complete [16]	NP-complete [11]		
cycles	NP-complete [16]	NP-complete [11]		
bounded paths	Polynomial [16]			
bounded cycles	Polynomial [16]			

2.1 $(\mathcal{X}, \mathcal{Y})$-Graphs in the Strong Model

Only for the strong model it makes sense considering the case when $\mathcal{X}$-graphs are general graphs, that is, when they are not constrained, since with such a hypothesis any graph is an $(\mathcal{X}, \mathcal{Y})$-graph in the weak model. Recognizing (general) graphs of cliques is known as Clique-Cover and is a renown NP-complete problem [8]. If the size of the cliques is fixed at 3, the problem transforms into recognizing graphs of triangles, also known as Partition-Into-Triangles, which is NP-complete [4, problem GT11]. Recognizing planar graphs of cliques is NP-hard [10, 9]. This problem stays NP-complete even if the planar graph is restricted to be a cycle of length three, although for cycles of length greater than 5 the problem is polynomial [3].

Table 1 summarizes known results when $\mathcal{Y}$-graphs are cliques in the strong model.

Similarly, regarding the families of $\mathcal{Y}$-graphs, we signal as important for applications the cases when $\mathcal{Y}$-graphs are "high-density" graphs, as, for example, graphs with high clustering coefficient; k-cores; cliques; complete bipartite graphs; k-connected graphs; strongly connected digraphs; and stars.

We recall that the clustering coefficient of a graph $G = (V, E)$ is defined as $\frac{2|E_i|}{|V_i|(|V_i|-1)}$, and that a graph has core k if all its vertices have degree at least k.

Again, from a more theoretic perspective, the cases when $\mathcal{Y}$-graphs are trees, paths, cycles, and bounded size graphs are interesting.

Table 2 summarizes known results in the strong model when $\mathcal{Y}$-graphs are paths or cycles. In [16] it is shown that deciding if it is possible to collapse paths or cycles in such a way that the reduced graph is a tree is NP-complete. Conversely, if the length of the paths (cycles) is bounded by a constant the problem is polynomial [16]. In [11] it is proved that, for a given integer $k \geq 2$, it is NP-complete to decide whether or not a graph

Table 3: Known results and open problems for the weak model.

	$\mathcal{X}$-graphs
$\mathcal{Y}$-graphs	planar graph
k-core graphs (max k)	$O((m+n)\log(n))$ [2]
h-cliques ($h \geq 5$)	NP-hard (Sect. 4)
h-cliques ($h \leq 4$)	open

is a path of length $k-1$ of paths (cycles), and that it is NP-complete to decide whether or not a graph is a k-star or a k-clique of paths (cycles).

In contrast, in [11] it is shown that k-graphs of paths (cycles) can be recognized in polynomial time when the inputs are restricted to graphs of bounded treewidth.

2.2 $(\mathcal{X}, \mathcal{Y})$-Graphs in the Weak Model

The weak model was introduced in [1, 2], where sparse $\mathcal{X}$-graphs and dense $\mathcal{Y}$-graphs were investigated for their effectiveness on hybrid visualization of large graphs. Refer to Table 3 for known results and open problems within this model.

The k-core of a graph G is the graph obtained from G by recursively removing vertices of degree less than k. Let n and m be the number of vertices and edges of G, respectively. In [2] it is shown that there exists an $O((n+m)\log(n))$ algorithm to find the maximum k such that the reduced graph obtained by collapsing each connected component of the k-core of G is planar.

In Sect. 4 we show that recognizing (planar, K_h)-graphs is NP-hard whenever $h \geq 5$. This result parallels the analogous result for the strong model [10, 9] and generalizes the result in [1], where only (planar, K_5)-graphs are considered.

The problem of recognizing (planar, K_h)-graphs remains open for $h \leq 4$.

3 Recognizing $(planar, K_h)$-graphs is in NP

We prove the following theorem.

Theorem 2 *Deciding whether a graph is a (planar,K_h)-graph, for $h \geq 5$, is in NP.*

Proof: A non-deterministic Turing machine to decide the problem can be devised as follows.

First, the set V of the vertices of the input graph $G(V, E)$ is non-deterministically partitioned in all possible ways into subsets $V_1, V_2, \ldots, V_q$. This can be performed in polynomial time and with a fixed degree of non-determinism by browsing the list of vertices and deciding non-deterministically to add the current vertex into the current subset V_i or to skip to the next vertex.

Second, the list of subsets is traversed in its turn and each V_i is non-deterministically either selected for contraction or not.

Third, each set V_i selected for contraction is checked, in order to assess whether it contains exactly h vertices and whether it actually induces a K_h subgraph. Only if both the tests are passed V_i is collapsed.

Finally, a standard planarity algorithm provides the answer for the current branch of the non-deterministic computation. □

We remark that the problem remains in NP even if we allow the contraction of any clique of size greater than or equal to h. This can be proved by slightly modifying the algorithm provided in the proof of Theorem 2.

4 Recognizing (planar, K_h)-graphs is NP-hard

Recognizing a planar graph of cliques is NP-hard in the strong model [10, 9]. In this section we show that (planar, K_h)-recognition is NP-hard also in the weak model for $h \geq 5$.

Theorem 3 *Deciding whether a graph is a (planar,K_h)-graph, for $h \geq 5$, is NP-hard.*

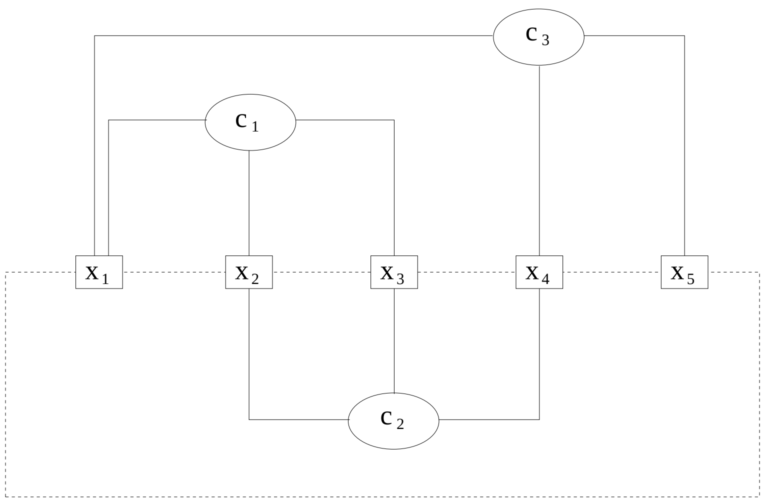


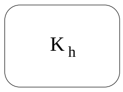
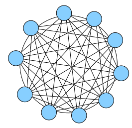
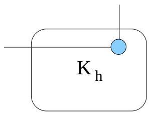
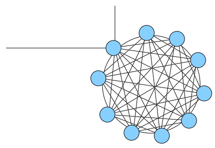
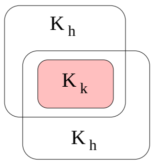
Figure 1: An instance of the Planar 3-Satisfiability (P3SAT) problem corresponding to the formula $(x_1 \vee x_2 \vee \overline{x}_3) \wedge (\overline{x}_2 \vee x_3 \vee \overline{x}_4) \wedge (\overline{x}_1 \vee \overline{x}_4 \vee x_5)$.

To prove the theorem we use a reduction from the Planar 3-Satisfiability problem [12]:

Problem: **Planar 3-Satisfiability (P3SAT)**

Instance: A set of clauses $c_1, \ldots, c_m$ each one having three literals from a set of Boolean variables $x_1, \ldots, x_n$. A plane bipartite graph $G(V_A, V_B, E)$ where nodes in V_A correspond to the variables while nodes in V_B correspond to the clauses (hence, $|V_A| = n$ and $|V_B| = m$). Edges connect clauses to the variables of the literals they contain. Moreover, $G(V_A, V_B, E)$ is drawn without intersections on a rectangular grid of polynomial size in such a way that nodes in V_A are arranged in a horizontal line that is not crossed by any edge (see Fig. 1).

Table 4: Keys of the symbols used in Figs 2–11.

Symbol	Description	
	A regular node	
	A K_h subgraph. Replaces:	
	A K_h subgraph with a notable node. Replaces:	
	Two K_h sharing k nodes	
	A collapsed K_h in the reduced graph	

Question: Can truth values be assigned to the variables $x_1, \ldots, x_n$ such that each clause has at least one true literal?

First, we describe how to construct the instance of the (planar,K_h)-recognition problem starting from an instance of the P3SAT problem. Second, we show that the P3SAT instance admits a solution if and only if the (planar,K_h)-recognition instance does.

Suppose I_{P3SAT} is an instance of the P3SAT problem with n Boolean variables and m clauses. The corresponding instance $I_{(planar,K_h)}$ is built by gluing together *variable gadgets* and *clause gadgets*, linked by *transmission gadgets* and *negation gadgets*. The construction rules for the aforementioned gadgets are detailed in the following sections. The keys of the symbols used in Figs 2–11 can be found in Table 4.

4.1 The Variable Gadget

The basic tool to construct the variable gadget is the *ring gadget* shown in Fig. 2 (Fig. 3(a) shows a section of it). Such a gadget is composed by an even number of intersecting K_h subgraphs, denoted $K_h^0, K_h^1, K_h^2, \ldots, K_h^p$, where p is an odd number. Subgraph K_h^i with i even (odd, respectfully) shares $\lceil (h-1)/2 \rceil$ nodes ($\lfloor (h-1)/2 \rfloor$ nodes, respectfully) with $K_h^{i+1 \mod p}$ (See Fig. 3(a)). Each K_h^i has exactly one node which is not shared with $K_h^{(i-1) \mod p}$ nor $K_h^{(i+1) \mod p}$. Such nodes are called *free nodes*. An edge joins the free node of K_h^i with that of $K_h^{(i+2) \mod p}$.

Two possible contractions of the ring gadget can produce a planar graph, corresponding to contracting all K_h^i with i even (see Fig. 3(b)) or to contracting all K_h^i with i odd

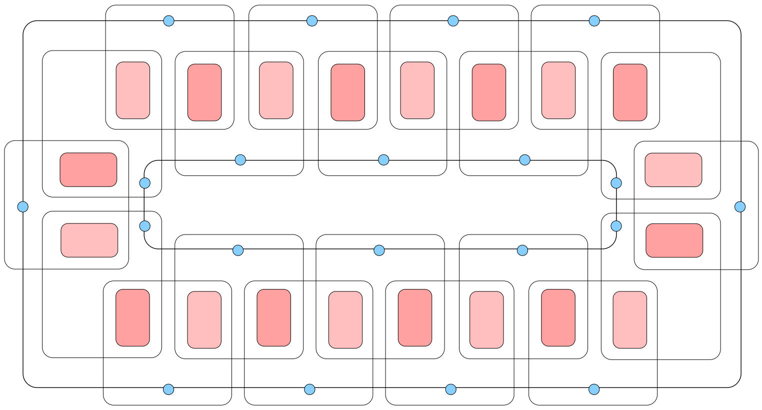


Figure 2: The *ring gadget*. White squares represent K_h subgraphs. Darker squares represent $K_{\lceil (h-1)/2 \rceil}$ subgraphs. Lighter squares represent $K_{\lfloor (h-1)/2 \rfloor}$ subgraphs (see also Fig. 3).

(see Fig. 3(c)). In these two cases we say that the chain gadget is *internally contracted* or *externally contracted*, respectively (see Fig. 4).

The following lemmas show that any other "mixed" contraction produces a non-planar graph. We start by showing that in any planar contraction of the ring gadget any three subsequent K_h, one must be contracted.

Lemma 1 *In any planar contraction of the ring gadget one among any three subsequent subgraphs K_h^i, K_h^{i+1}, and K_h^{i+2} is contracted if $h \geq 5$.*

Proof: Suppose for contradiction that there exists a planar contraction such that no K_h is contracted among K_h^i, K_h^{i+1}, and K_h^{i+2}. Since K_h^{i+1} only shares vertices with K_h^i and K_h^{i+2}, all vertices and edges of K_h^{i+1} will be in the planar graph. As $h \geq 5$ the contracted graph can not be planar and we have a contradiction. □

Observe that in any planar contraction of the ring gadget two subsequent subgraphs K_h^i and K_h^{i+1} can not be simultaneously contracted (since they share some nodes). On the other hand, we show that for any pair of subsequent K_h^i and K_h^{i+1}, one must be contracted.

Lemma 2 *In any planar contraction of the ring gadget one among two subsequent subgraphs K_h^i and K_h^{i+1} is contracted if $h \geq 5$.*

Proof: Consider a sequence of $K_h^i, K_h^{i+1}, K_h^{i+2}$, and K_h^{i+3}. Assume for a contradiction that there exists a planar contraction of the ring gadget such that K_h^{i+1} and K_h^{i+2} are left not contracted. By Lemma 1 applied to K_h^i, K_h^{i+1}, and K_h^{i+2}, we have that K_h^i is contracted. The same lemma applied to K_h^{i+1}, K_h^{i+2}, and K_h^{i+3}, gives that K_h^{i+3} is contracted. Two are the cases: either i is even or is odd. Suppose i is even. This implies that a $K_{\lfloor (h-1)/2 \rfloor}$ is not involved in any contraction (see Fig. 5(a)) and allows to find a K_5 subdivision in the contracted graph if $\lfloor (h-1)/2 \rfloor \geq 2$, i.e., if $h \geq 5$ (see Fig. 5(b)) contradicting the hypothesis that the contracted graph is planar. Now, suppose i is odd. A $K_{\lceil (h-1)/2 \rceil}$ is not involved in any contraction, and this leaves a K_5 in the contracted graph whenever $h \geq 4$ (see Fig. 5(c)) yielding again a contradiction. □

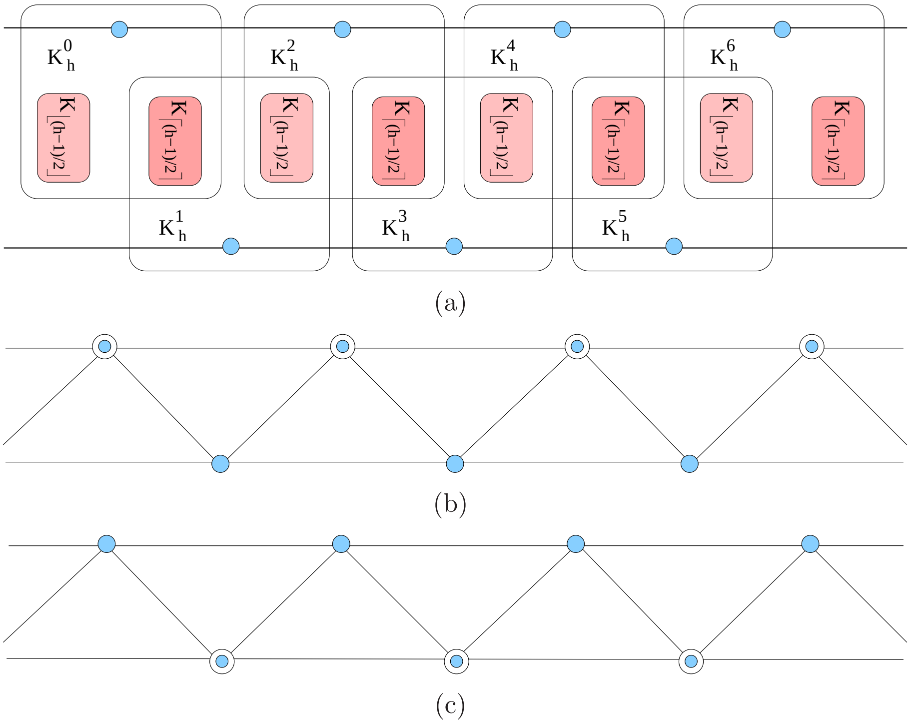


Figure 3: (a) A section of the *ring gadget*. All K_h^i, with i even (b), or all K_h^l, with l odd (c) can be contracted to yield a planar graph.

Lemma 3 *Let $h \geq 5$ be an integer. In any planar contraction of the ring gadget either the ring gadget is internally contracted (all K_h^i, with i odd, are contracted) or it is externally contracted (all K_h^l, with l even, are contracted).*

Proof: Since $h \geq 5$ Lemma 2 holds and in any planar contraction of the ring gadget, one among any two subsequent K_h subgraphs is contracted. On the other hand, two subsequent K_h can not be simultaneously contracted. It follows that contracted and non-contracted K_h alternate along the ring gadget. □

The *variable gadget* is simply a ring gadget. The planar sub-graph obtained by internally contracting the ring gadget is associated with a `true` value for the corresponding variable, while the planar sub-graph obtained by externally contracting the ring gadget is associated with a `false` value. Observe that the shape of the variable gadget can be

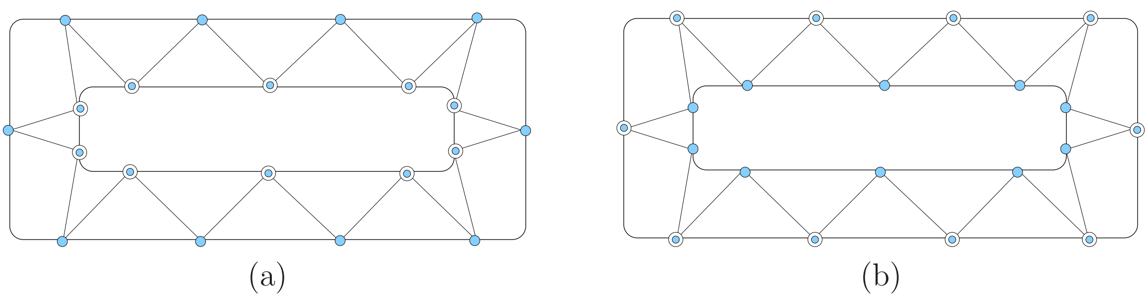


Figure 4: The *chain gadget* can be *internally contracted* (a) or *externally contracted* (b).

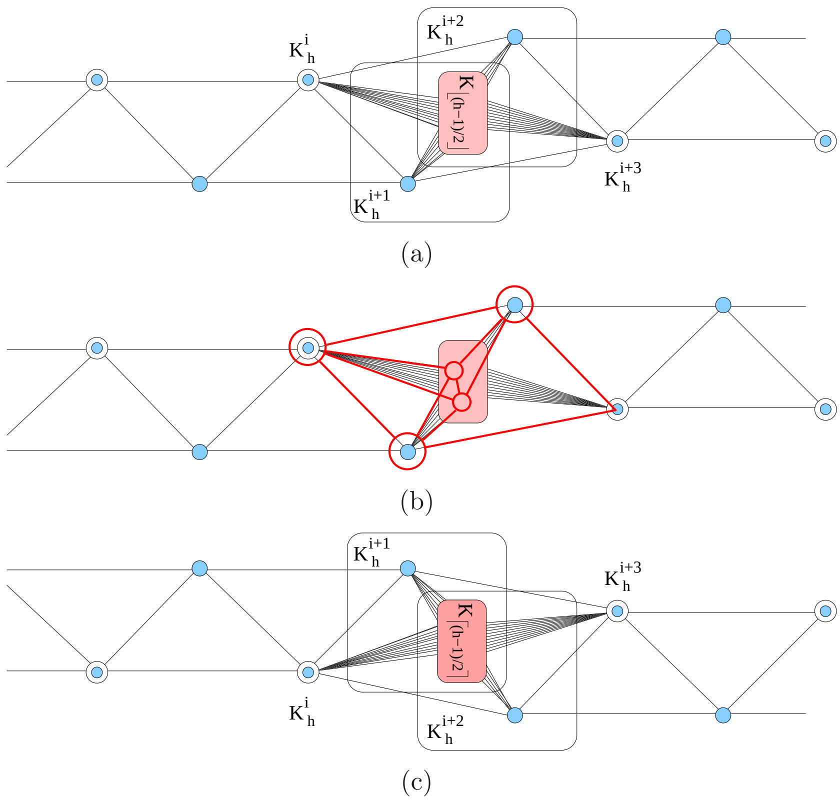


Figure 5: A figure for the proof of Lemma 2. (a) The ring gadget after a contraction in the case in which i is even. (b) A K_5 subdivision can be found in the contracted graph if $\lfloor(h-1)/2\rfloor \geq 2$ (i.e., if $h \geq 5$). (c) The ring gadget after a contraction in the case in which i is odd.

modified in order to reach all the clause gadgets that contain a literal of the corresponding variable (see Fig. 6). Also, the variable gadgets of different variables can be linked together to form a line (again, see Fig. 6).

4.2 The Negation Gadget

The *negation gadget* is used to attach a variable gadget to a clause gadget when the corresponding literal is negated. Details on where to place negation gadgets are given in Section 4.3 where clause gadgets are described. The negation gadget, shown in Fig. 7(a), consists of a chain gadget suitably attached to the variable gadget. Namely, a K_h^i, with i odd, is chosen in the variable gadget and a K_h^j, with j also odd, is chosen in the added chain gadget. The free node v_i of K_h^i is identified with the free node v_j of K_h^j. Also, an edge is added between an arbitrary node $v_i' \neq v_i$ of K_h^i and an arbitrary node $v_j' \neq v_j$ of K_h^j. Finally, it is assumed that there exists a path connecting the variable gadget with the negation gadget which does not use $v_i = v_j$ nor the edge (v_i', v_j') (dashed line of Fig. 7(a)).

Lemma 4 *Let $h \geq 5$ be an integer and let I_h be an instance of (planar,K_h)-recognition containing a variable gadget attached to a negation gadget. In any planar contraction of I_h all K_h^i of the variable gadget, with i odd, are contracted if and only if all K_h^l of the negation gadget, with l even, are contracted.*

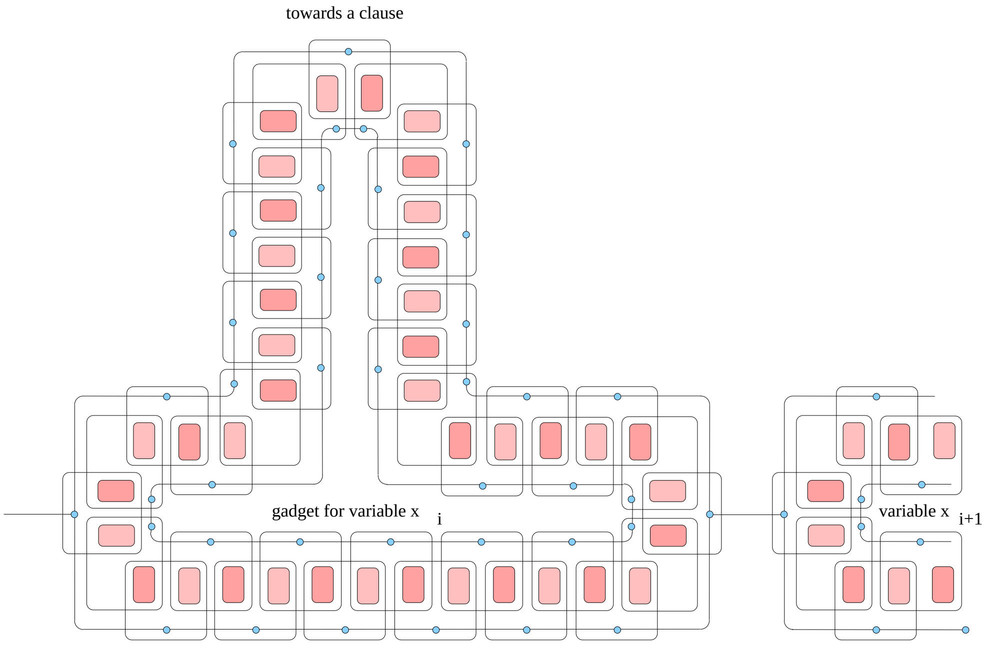


Figure 6: The *junction gadget*.

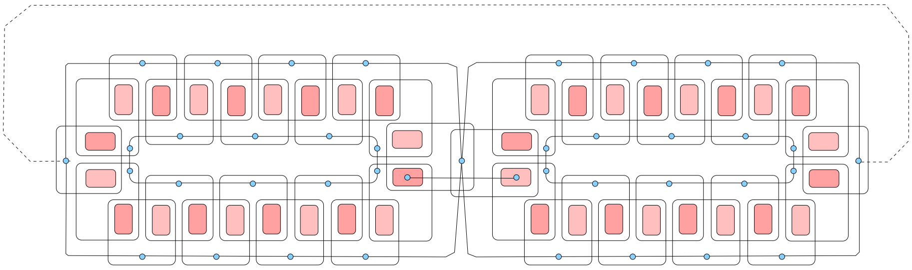
Proof: Due to Lemma 3 the variable gadget and the negation gadget are each internally or externally contracted. Figure 7(b) shows that if the variable gadget is externally contracted (i.e., all its K_h^l, with l odd, are contracted) and the negation gadget is internally contracted (i.e., all its K_h^m, with m even, are contracted) the resulting graph is planar. Analogously, a mirrored image of Fig. 7(b) can be used to show that internally contracting the variable gadget and externally contracting the negation gadget yields a planar graph. It is obvious that both the variable gadget and the negation gadget can not be both externally contracted (this is because K_h^i and K_h^j share a vertex, with i and j odd). Figure 7(c) shows that the converse also yields non-planarity, i.e., that both the variable gadget and the negation gadget can not have all K_h^i contracted with i even. In fact, in that case a $K_{3,3}$ subdivision can be found in the contracted graph. □

4.3 The Clause Gadget

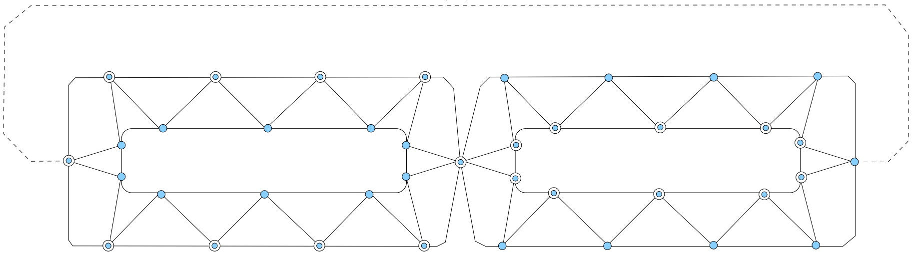
Consider a clause with three literals $(l_1 \wedge l_2 \wedge l_3)$ from three Boolean variables x_1, x_2, and x_3. If l_i, with $i \in \{1, 2, 3\}$, is a positive literal of variable x_i we directly attach the variable gadget for x_i to the clause gadget, otherwise we insert a negation gadget between the variable gadget for x_i and the clause gadget.

Figure 8 shows the construction for the clause gadget. The clause gadget consists of a K_{h+2} subgraph which shares a node with the variable gadget (or the negation gadget) from each variable of the clause.

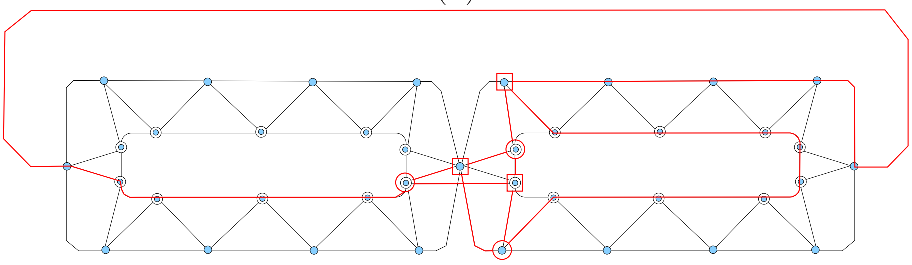
Lemma 5 *Let $h \geq 5$ be an integer and let I_h be an instance of (planar,K_h)-recognition containing a clause gadget for clause $(l_1 \wedge l_2 \wedge l_3)$. The clause gadget can be contracted to yield a planar graph if and only if at least one of the rings is internally contracted.*



(a)



(b)



(c)

Figure 7: (a) The *negation gadget* is composed by two variable gadgets.

Proof: If literal l_i is `false`, the corresponding variable gadget (or negation gadget) is externally contracted. If all three literals are false (see Fig. 9) the non-contracted K_{h-1} and the three contracted nodes form a K_{h+2}, which determines non-planarity whenever $h \geq 3$.

Conversely, if at least one literal l_i is `true` (see Fig. 10), then the corresponding variable gadget (or negation gadget) is internally contracted, and the node shared with the clause gadget is not involved in any contraction. Hence, a K_{h-1} can be found and contracted, producing a triangle in the reduced graph (see Fig. 11). □

4.4 Proof of Theorem 3

By using the gadgets given in the previous sections, starting from an instance I_{P3SAT} of the P3SAT problem we build an instance $I_{(planar,K_h)}$ of (planar,K_h)-recognition in polynomial time. Based on such a construction Theorem 3 is proved by showing that the I_{P3SAT} instance admits a solution if and only if the $I_{(planar,K_h)}$ instance admits one.

Namely, given a truth assignment for the Boolean variables that satisfies instance I_{P3SAT}, we decide the state of each variable gadget by internally contracting (externally contracting, respectively) its ring gadget if the variable is `true` (`false`, respectively). The truth values propagate from the variable gadgets towards the clause gadgets possibly

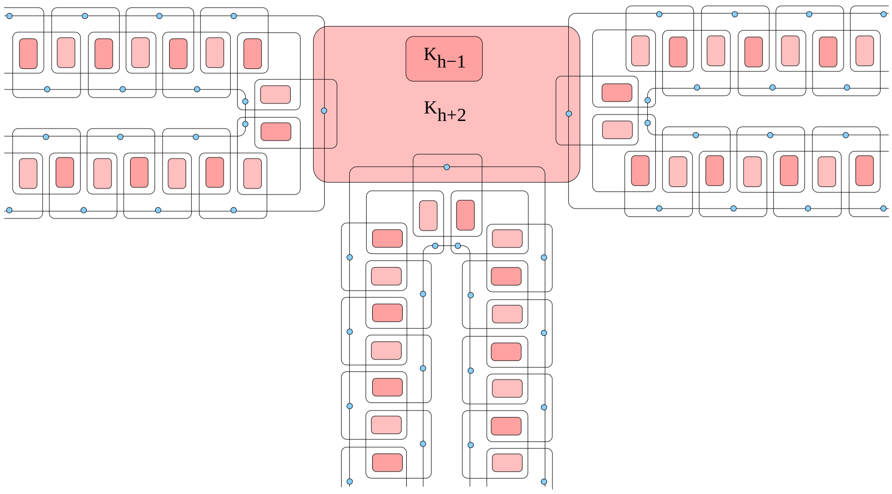


Figure 8: The *clause gadget*. For a positive literal the variable gadget directly intersect the clause gadget. For a negative literal a negation gadget is placed between the variable gadget and the clause gadget. Hence, if a literal is false, the clause gadget is intersected by an externally contracted ring.

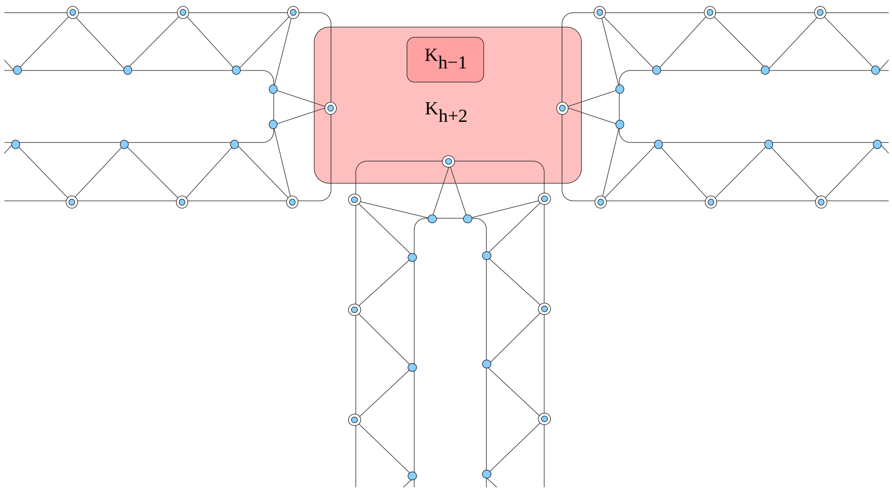


Figure 9: When all literals are `false` there can not be any further contraction in the clause gadget and a K_{h+2} yields non-planarity for the reduced graph.

traversing a negation gadget. Since each clause has at least a true literal, each clause gadget is reached at least by one internally-contracted ring gadget and Lemma 5 ensures that the clause gadget can be contracted into a planar subgraph.

Conversely, suppose that $I_{(planar,K_h)}$ yields a planar graph when certain K_h's are collapsed. By Lemma 5 each clause gadget has at least a `true` literal and the corresponding variable gadget is coherently contracted. Hence, a truth assignment can be determined for the Boolean variables and such an assignment satisfies instance I_{P3SAT}.

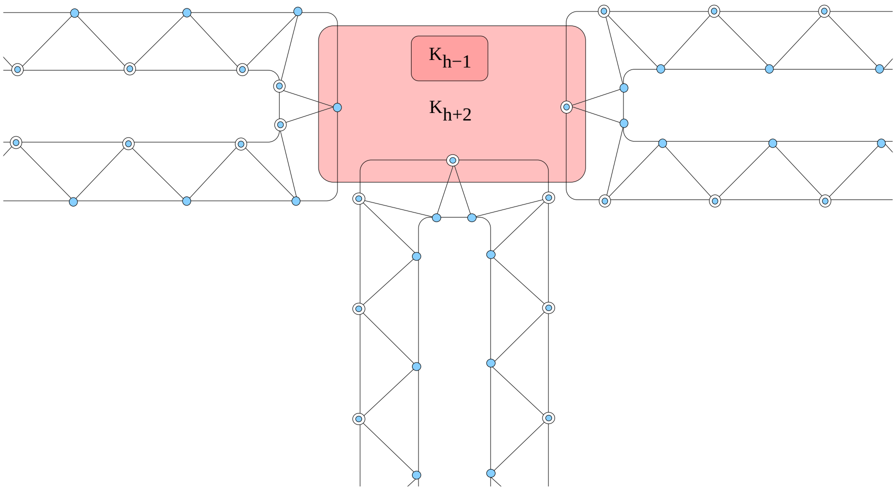


Figure 10: The clause gadget when one literal is `true` (the one on the left side).

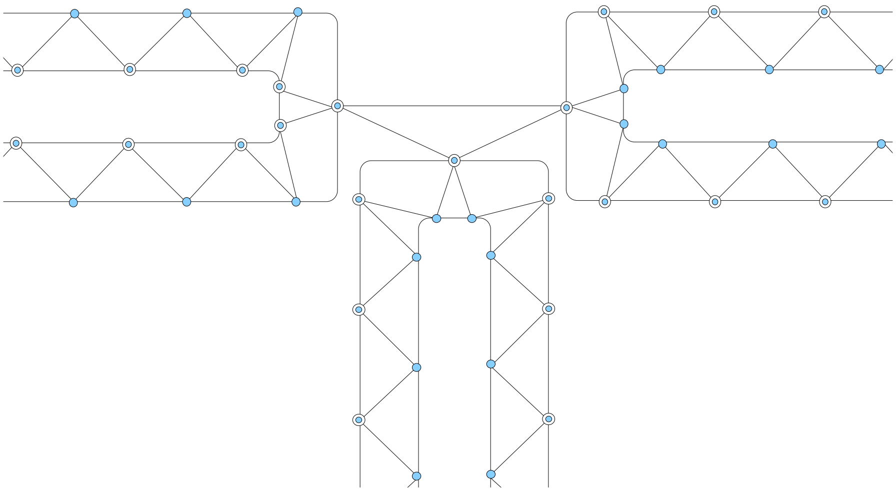


Figure 11: When at least a literal is `true`, a further contraction transforms the clause gadget into a triangle.

5 Conclusions and Final Remarks

We showed that, for any $h \geq 5$, it is NP-complete to decide whether or not a graph $G(V, E)$ is a planar graph of K_h in the weak model (i.e., when the clusters are not requested to be a partition of V). This result parallels the analogous result for the strong model [10, 9] and generalizes the result in [1], where only (planar, K_5)-graphs are considered.

We remark that both the algorithm in the proof of Theorem 2 and the reduction used to prove Theorem 3 would hold if we allow the contraction of any clique of size greater than or equal to $h \geq 5$. Hence, even collapsing cliques of heterogeneous sizes to obtain a planar graph is an NP-complete problem if the sizes of the cliques to collapse are requested to be greater than or equal to five.

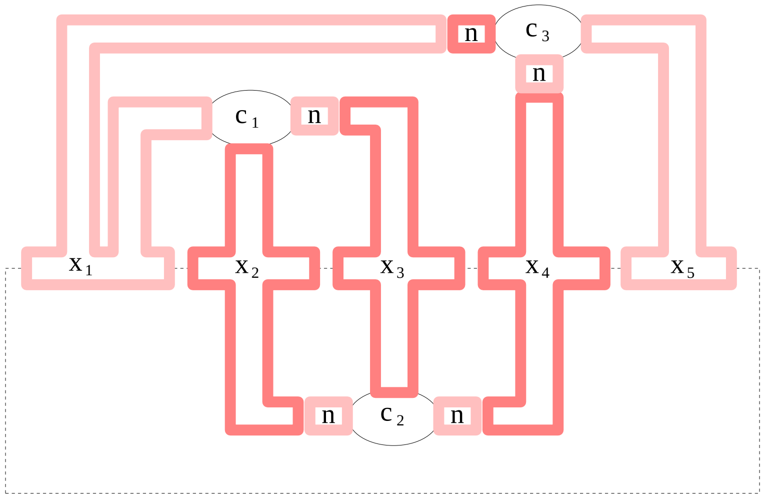


Figure 12: The same instance of PLANAR 3SAT of Fig. 1 corresponding to the formula $(x_1 \vee x_2 \vee \overline{x}_3) \wedge (\overline{x}_2 \vee x_3 \vee \overline{x}_4) \wedge (\overline{x}_1 \vee \overline{x}_4 \vee x_5)$. Darker rings are internally contracted. The contracted graph corresponds to the truth assignment $x_1 = x_5 =$ `false` and $x_2 = x_3 = x_4 =$ `true`.

We leave as open the problem of recognizing (planar, K_h)-graphs with $h \leq 4$.

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