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On Agreement Problems with Gossip Algorithms in absence of common reference frames

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ABSTRACT

In this paper a novel approach to the problem of decentralized agreement toward a common point in space in a multi-agent system is proposed. Our method allows the agents to agree on the relative location of the network centroid respect to themselves, on a common reference frame and therefore on a common heading. Using this information a global positioning system for the agents using only local measurements can be achieved. In the proposed scenario, an agent is able to sense the distance between itself and its neighbors and the direction in which it sees its neighbors with respect to its local reference frame. Furthermore only point-to-point asynchronous communications between neighboring agents are allowed thus achieving robustness against random communication failures. The proposed algorithms can be thought as general tools to locally retrieve global information usually not available to the agents.

1 Introduction

In the last decade networked multi-agent systems have drawn the attention of a large part of the control systems community. Most of the attention has been devoted to the development of decentralized motion coordination algorithms, for a representative example see [7, 11, 12, 19, 17]. In this framework, coordination algorithms have been developed making use of either absolute position information or relative distance measurements between agents [13, 8, 14, 5] to perform the most various tasks, for instance rendezvous [3], leader following [18], attitude control [4] and many others [9, 10, 16].

Many of these algorithms, dealing with decentralized motion coordination problems, assume that the agents have access to absolute position information (GPS) and thus have a common global reference frame that makes it easy to interpret the information passed by other agents. Even when in multi agent systems the agents are not supposed to know their absolute position, many times they are assumed to have a common attitude reference to exchange information that can be achieved by using a compass and gravity as common reference for their coordinate system. For space applications another technological solution is to use a frame of fixed stars to have a common reference. In all these instances several technological countermeasures have to be undertaken for the implementation of coordination algorithms increasing the total costs of the single agents. On the other hand, algorithms that use only relative distance measurements tend to achieve only low complexity tasks due to extremely difficult coordination problems in absence of reference frames.

We believe that having a common reference frame, or agreement on some common fixed points in space, greatly simplifies the necessary coordination algorithms and increases their effectiveness. Nevertheless, not relaying on external systems like GPS could significantly advance the technological feasibility of mobile swarms of agents, reducing their dependence on the global positioning system in the low level control loops.

In this paper a novel approach to the problem of decentralized agreement toward a common point in space in a multi-agent system is proposed. The proposed method allows the agents to agree on the network centroid, on a common reference frame, on a common heading. Using this information a global positioning system for the agents using only local measurements can be achieved. Furthermore only point-to-point asynchronous communications between neighboring agents are allowed thus achieving robustness against random communication failures. The proposed algorithms can be thought as general tools to locally retrieve global information usually not available to the agents. In this way, any assumption on the absence of a common reference frame could be relaxed and therefore, simpler algorithms could be developed.

2 Background on Gossip algorithms over networks

Let the network of agents be described by a time-varying graph $\mathcal{G}(t) = \{V, E(t)\}$, where $V = \{v_i : i = 1, \ldots, n\}$ is the set of nodes (agents) and $E(t) = \{e_{ij} = (v_i, v_j)\}$ is the set of edges (connectivity) representing the point-to-point communication channel availability at time t. A position $p_i \in \mathbb{R}^d$ in the d^{th} dimensional space is associated to each node $v_i \in V$, with $i = 1, \ldots, n$. In particular, an edge representing a connection between two agents exists if and only if the distance between these agents is less then or equal to their sensing radius r, namely

$$E(t) = \{e_{ij} : \ \|p_i(t) - p_j(t)\| \leq k, \ i \neq j\},$$

where $\|\cdot\|_d$ is the Euclidean norm in $\mathbb{R}^d$. Therefore, a generic couple of agents $\{i, j\}$ is able to sense $\|p_i - p_j\|$ reciprocally. In addition, each agent has a local reference frame defined by an orthonormal basis of vectors in $\mathbb{R}^d$ fixed on it and, is able to determine the direction in which neighbors are sensed, strictly with respect to its own local reference frame.

In the proposed framework a gossip algorithm is defined as a triplet $\{\mathcal{S}, \mathcal{R}, \mathfrak{e}\}$ where

- $\mathcal{S} = \{s_1, \ldots, s_n\}$ is a set containing the local estimate s_i of each agent i in the network.
- $\mathcal{R}$ is a local interaction rule that given edge e_{ij} and the states of agents i, j $\mathcal{R} : (s_i, s_j) \Rightarrow (\hat{s}_i, \hat{s}_j)$.
- $\mathfrak{e}$ is a edge selection process that specifies which edge $e_{ij} \in E(t)$ is selected at time t.

From an algorithmic point of view, a possible implementation of the gossip algorithm described above is given in Algorithm 1.

Algorithm 1: Gossip Algorithm

Data: $t = 0$, $s_i(0) = s_{i0} \quad \forall\, i = 1, \ldots, n$.
Result: $s_i(t_{stop}) \quad \forall\, i = 1, \ldots, n$.

while *stop_condition* **do**

- Let $t = t + 1$.
- Select an edge $e_{ij} \in E(t)$ according to $\mathfrak{e}$.
- Update the states of the selected agents applying $\mathcal{R}$:

$$(s_i(t+1), s_j(t+1)) = \mathcal{R}(s_i(t), s_j(t)).$$

end

Definition 1 *Let us define* $\mathbb{G}(t, t+\Delta t) = \{V, \mathbb{E}(t, t+\Delta t)\}$*, where* $\mathbb{E}(t, t+\Delta t) = \bigcup_{k=t}^{t+\Delta t} \mathfrak{e}(k)$*, as the graph resulting from the union of all the edges given by the edge selection process from time* t *to* $t + \Delta t$*.*

3 Problem description

Let us consider a network of agents with limited sensing capabilities. Each agent, which is characterized by a position in a 2-dimensional space, is able to cooperate with its neighboring agents, i.e., agents that are within its sensing radius. The following assumptions on the network of agents are made:

Assumptions 1

- *The network can be described by a connected undirected switching graph.*
- *Sensing range is limited by a maximum sensing radius r.*
- *Communications are asynchronous,* gossip *like [2].*
- *Each node can sense the distance between itself and its neighbors.*
- *Each node can sense the direction in which it sees its neighbors with respect to its local reference frame, arbitrary fixed on it.*

■

Note that, for each agent i it is possible to express its estimate s_i with respect to a global reference frame by introducing a rotation matrix R_i as follows:

$$s_{gi} = R_i s_i + p_i. \tag{1}$$

Our first objective is to make the local estimate of each agent converge to a common value by applying an iterative algorithm so that:

$$\forall i, \quad \lim_{t\to\infty} s_{gi}(t) = R_i s_i(t) + p_i = \frac{1}{n}\sum_{i=1}^{n}\Big(R_i s_i(0) + p_i\Big).$$

4 Agreement on a common point in a 2-D space

In this section, we specify an interaction rule $\mathcal{R}$ such that an agreement on a common point under assumptions 1 is achieved.

Given a couple of nodes $\{i, j\}$ for which an edge exists, that is $e_{ij} \in E(t)$, let us define the direction for which node i is able to sense node j with respect to its local frame as

$$\hat{c}_{ij} = R_i^T \frac{(p_j - p_i)}{\|p_j - p_i\|},$$

where $p_i, p_j \in \mathbb{R}^2$. Clearly, the following property holds $R_i c_{ij} = -R_j c_{ji}$. Furthermore we define the orthogonal versor $\hat{c}_{ij}$ so that a right handed frame is built.

In addition, let the relative distance between two nodes i and j be:

$$d_{ij} = d_{ji} = \|p_i - p_j\|_2.$$

Finally, let the network of agents be deployed in a 2-dimensional space. The proposed algorithm consists of an edge selection process $\mathfrak{e}$ that specifies which edge $e_{ij} \in E(t)$ is active at time t and a local interaction rule $\mathcal{R}$ that specifies how to update the estimates of agents v_i and v_j.

Now follows the definition of $\mathcal{S}$ and $\mathcal{R}$:

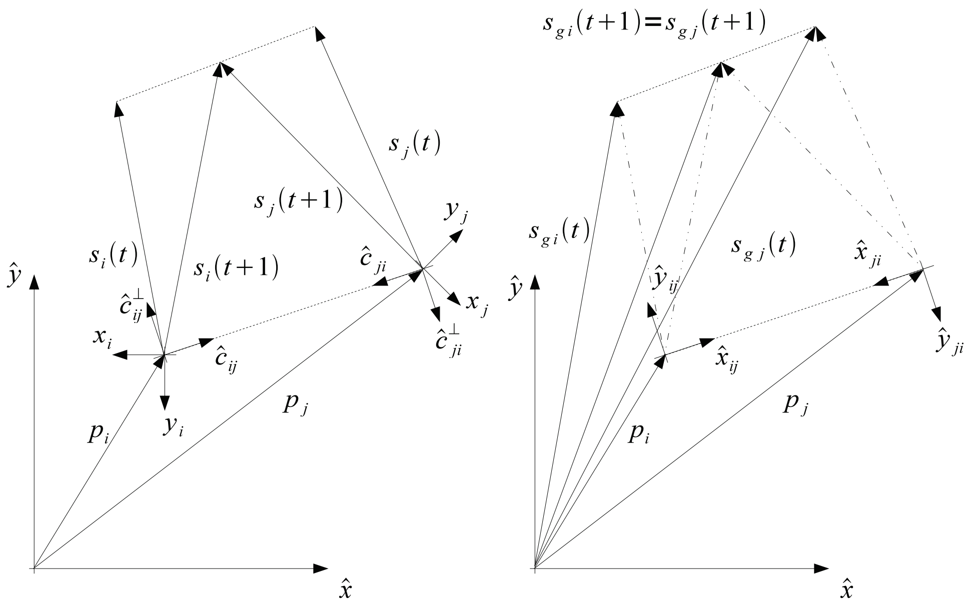

Figure 1: Example of algorithm iteration involving two nodes. a) On the left, with respect to the agents local reference frames. b) On the right, with respect to a global reference frame.

Definition 2 ($\mathcal{S}$)
Let $\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$, *with* $s_i \in \mathbb{R}^2, \forall i = 1, \ldots, n$ *be the set of current agents local estimates, each one in their own reference frame.* ■

Definition 3 ($\mathcal{R}$)

- *Let*

$$\begin{aligned} \Delta(t) &= \frac{d_{ij} - s_j(t)^T \hat{c}_{ji} + s_i(t)^T \hat{c}_{ij}}{2}, \\ \Delta^{\perp}(t) &= \frac{s_i(t)^T \hat{c}_{ij}^{\perp} - s_j(t)^T \hat{c}_{ji}^{\perp}}{2}, \end{aligned} \tag{2}$$

- $\mathcal{R}$*:*

$$\begin{aligned} s_i(t+1) &= \Delta(t) \cdot \hat{c}_{ij} + \Delta^{\perp}(t) \cdot \hat{c}_{ij}^{\perp}, \\ s_j(t+1) &= \Delta(t) \cdot \hat{c}_{ji} + \Delta^{\perp}(t) \cdot \hat{c}_{ji}^{\perp}. \end{aligned} \tag{3}$$

■

As a support for the algorithm description, Fig. 1 depicts a possible scenario involving two nodes, namely i and j.

A couple of remarks are now in order:

- This update rule leads itself to an easy decentralized implementation of the algorithm,

- All the parameters are local to the agents and independent to any specific reference frame as they rely on a common direction given by the line of sight between the two agents.
- The two selected agents estimate the relative position between each other, namely (in the following with respect to agent i, the same holds for agent j) their distance $\|p_i - p_j\|$ and the line of sight $\hat{c}_{ij}$ both in their own local reference frame. They then compute the projection of their current estimate with respect to the line of sight $s_i(t)^T\,\hat{c}_{ij}$ and the perpendicular to it $s_i(t)^T\,\hat{c}_{ij}^{\perp}$ between them and transmit this scalar value to their companion.
- The two agents then update their estimates independently by averaging between their projections on the line of sight and updating their estimate along the direction of the line of sight.
- The proposed gossip algorithm allows to converge to a common point in a 2-dimensional space. As it will be shown, the convergence to the centroid of the network is simply a consequence of the particular choice of the initial conditions.

The following Lemma shows that the proposed gossip algorithm can be equivalently stated with respect to a global common reference frame. Indeed, this will be exploited in the rest of the paper to investigate its convergence properties.

Lemma 1 *The gossip algorithm $\{\mathcal{S}, \mathcal{R}, \mathfrak{e}\}$, with $\mathcal{S}, \mathcal{R}$ defined respectively as in (2), (3) can be equivalently stated with respect to a global common reference frame as follows:*

$$
\begin{aligned}
x(t+1) &= W(\mathfrak{e}(t))x(t),\\
y(t+1) &= W(\mathfrak{e}(t))y(t),
\end{aligned}
$$

where $W(\mathfrak{e}(t))$ is a matrix representation of the update rule $\mathcal{R}$, $x(t) = [x_1(t),\, \dots,\, x_n(t)]^T \in \mathbb{R}^n$, $y(t) = [y_1(t),\, \dots,\, y_n(t)]^T \in \mathbb{R}^n$, and $s_{gi}(t) = [x_i(t)\; y_i(t)]^T$.

***Proof* 1** *Let us consider a generic update for a couple of agents $\{i, j\}$. Given the estimates $(s_i(t), s_j(t))$ at time t, according to the rule $\mathcal{R}$ given in (3) the estimates at time $t+1$ would be:*

$$
\begin{aligned}
s_i(t+1) = &\quad \Delta(t)\cdot\hat{c}_{ij} + \Delta^{\perp}(t)\cdot\hat{c}_{ij}^{\perp},\\
s_j(t+1) = &\quad \Delta(t)\cdot\hat{c}_{ji} + \Delta^{\perp}(t)\cdot\hat{c}_{ji}^{\perp}.
\end{aligned}
$$

Now by substituting the Δ according to the definition given in (2) we have:

$$
\begin{aligned}
s_i(t+1) &= \frac{1}{2}\left(d_{ij} - s_j(t)^T\hat{c}_{ji} + s_i(t)^T\hat{c}_{ij}\right)\cdot\hat{c}_{ij} + \\
&+\frac{1}{2}\left(s_i(t)^T\hat{c}_{ij}^{\perp} - s_j(t)^T\hat{c}_{ji}^{\perp}\right)\cdot\hat{c}_{ij}^{\perp}, \\
s_j(t+1) &= \frac{1}{2}\left(d_{ij} - s_j(t)^T\hat{c}_{ji} + s_i(t)^T\hat{c}_{ij}\right)\cdot\hat{c}_{ji} + \\
&+\frac{1}{2}\left(s_i(t)^T\hat{c}_{ij}^{\perp} - s_j(t)^T\hat{c}_{ji}^{\perp}\right)\cdot\hat{c}_{ji}^{\perp}.
\end{aligned}
$$

At this point, let us consider the update of the agent i with respect to a global frame as given in (1):

$$
\begin{aligned}
s_{gi}(t+1) &= R_i\frac{1}{2}\left(d_{ij} - s_j(t)^T\hat{c}_{ji} + s_i(t)^T\hat{c}_{ij}\right)\cdot\hat{c}_{ij} \\
&+\frac{1}{2}R_i\,\left(s_i(t)^T\hat{c}_{ij}^{\perp} - s_j(t)^T\hat{c}_{ji}^{\perp}\right)\cdot\hat{c}_{ij}^{\perp} + p_i, \\
&= \frac{1}{2}\left(d_{ij} - s_j(t)^T\hat{c}_{ji} + s_i(t)^T\hat{c}_{ij}\right)\cdot\hat{x}_{ij} + \\
&+\frac{1}{2}\left(s_i(t)^T\hat{c}_{ij}^{\perp} - s_j(t)^T\hat{c}_{ji}^{\perp}\right)\cdot\hat{y}_{ij} + p_i,
\end{aligned}
$$

where $[\hat{x}_{ij}, \hat{y}_{ij}]^T$ are the equivalent of $[\hat{c}_{ij}, \hat{c}_{ij}^{\perp}]^T$ in a common global reference frame. At this point, with respect to the local reference frame of agent i we have that:

$$
\begin{aligned}
x_{s_i} &= s_i(t)^T\hat{c}_{ij}, & y_{s_i} &= s_i(t)^T\hat{c}_{ij}^{\perp}, \\
x_{s_j} &= d_{ij} - s_j(t)^T\hat{c}_{ji}, & y_{s_j} &= -s_j(t)^T\hat{c}_{ji}^{\perp},
\end{aligned}
$$

which allows to re-write the previous equation as follows:

$$
\begin{aligned}
s_{gi}(t+1) &= \left(\frac{x_{s_i} + x_{s_j}}{2}\right)\cdot\hat{x}_{ij} + \left(\frac{y_{s_i} + y_{s_j}}{2}\right)\cdot\hat{y}_{ij} + p_i, \\
&= \frac{x_{s_i}\,\hat{x}_{ij} + y_{s_i}\,\hat{y}_{ij} + p_i}{2} + \frac{x_{s_j}\,\hat{x}_{ij} + y_{s_j}\,\hat{y}_{ij} + p_i}{2}, \\
&= \frac{s_{gi}(t) + s_{gj}(t)}{2}.
\end{aligned}
$$

Therefore according to the updating rule $\mathcal{R}$ given in (3) with respect to a global reference frame we have:

$$
\begin{aligned}
s_{gi}(t+1) &= \frac{s_{gi}(t) + s_{gj}(t)}{2}, \\
s_{gj}(t+1) &= \frac{s_{gi}(t) + s_{gj}(t)}{2}.
\end{aligned}
$$

Hence, we can decouple the coordinate system and study the evolution of the states in the two different axes separately:

$$
\begin{aligned}
x(t+1) &= W(\mathfrak{e}(t))x(t), \\
y(t+1) &= W(\mathfrak{e}(t))y(t),
\end{aligned}
$$

where, if at time t edge $e_{ij} = (i, j)$ is selected, we have:

$$W(e_{ij}) = I - \frac{(\hat{e}_i - \hat{e}_j)\,(\hat{e}_i - \hat{e}_j)^T}{2},$$

where $\hat{e}_i = [0\ldots 0 \underbrace{1}_{i} 0 \ldots 0]^T$ is a $n \times 1$ vector with all the components equal to 0 but the i-th component equal to 1.

Now, some technicalities are introduced. In particular, let us define the set of fixed points as:

$$C(e_{ij}) = \text{Fix}\, W(e_{ij}) = \{x \in \mathbb{R}^n : W(e_{ij})\, x = x\}$$

and the intersection $\hat{C}_{(t,t+\Delta t)}$ of the set of fixed points over time $[t, t+\Delta t]$ as:

$$\hat{C}_{(t,t+\Delta t)} \;=\; \bigcap_{e_{ij} \in \mathbb{E}(t,t+\Delta t)} C(e_{ij}).$$

Finally, let us define the quasi projection of x_0 onto C [1] as:

$$\mathcal{Q}_c\, x_0 = \{x \in C : \|x - c\| \leq \|x_0 - c\|, \forall c \in C\}.$$

The following Lemma states that if the graph representing the union of the selected edges is connected over a window of time, then the space representing the intersection of the images of the matrices corresponding to those edges is span$\{\mathbf{1}_n\}$.

Lemma 2 *If $\mathfrak{e}$ is such that $\forall t,\ \exists\, \Delta t\ : \mathbb{G}(t, t+\Delta t)$ is connected, then:*

$$\hat{C}_{(t,t+\Delta t)} \;=\; \bigcap_{e_{ij} \in \mathbb{E}(t,t+\Delta t)} C(e_{ij}) = \mathit{span}\{\mathbf{1}_n\}, \tag{4}$$

where $\mathbf{1}_n = [1,\, \ldots, 1]^T$ is a $n \times 1$ unit vector with all the components equal to 1.

Proof: See the Appendix A.1. □

To link the connectivity of the graph representing the union of the selected edges to the contractive property respect to span$\{\mathbf{1}_n\}$ of the product of the paracontracting matrices $W(e_{ij})$, the following lemma is needed:

Lemma 3 *If $\mathfrak{e}$ is such that $\forall t,\ \exists\, \Delta t\ : \mathbb{G}(t, t+\Delta t)$ is connected, then there exists a norm such that:*

$$\begin{gathered} \|W(e_{ij})\, x - c\| \leq \|x - c\|, \\ \forall\, c \in \hat{C}_{(t,t+\Delta t)}, \qquad \forall\, e_{ij} \in \mathbb{E}_{(t,t+\Delta t)},\ \forall\, x \in \mathbb{R}^n \end{gathered} \tag{5}$$

$$\|\Phi_{(t,t+\Delta t)}\, x - c\| < \|x - c\|, \qquad \forall\, c \in \hat{C}_{(t,t+\Delta t)}, \qquad \forall\, x \in \mathbb{R}^n \backslash \hat{C}_{(t,t+\Delta t)} \tag{6}$$

where $\Phi_{(t,t+\Delta t)} = \prod_{e_{ij} \in \mathbb{E}(t,t+\Delta t)} W(e_{ij})$.

Proof: See the Appendix A.2. □

In above Lemma 3 it is shown that the agents estimates eventually contract toward span$\{\mathbf{1}_n\}$. In the following Lemma it is shown that the trajectories of the system actually converge to some point in span$\{\mathbf{1}_n\}$.

Lemma 4 *If $\mathfrak{e}$ is such that $\forall t,\ \exists\ \Delta t\ : \mathbb{G}(t, t + \Delta t)$ is connected, then for any sequence of intervals $\{l_i\}$ where $l_i = l_{i-1} + \Delta t_i$ with $l_0 = 0$ and $l_j > l_i\ \forall\, j > i$, it holds:*

$$d\left(x(l_i), span\{\mathbf{1}_n\}\right) \to 0. \tag{7}$$

Proof: See the Appendix A.3. □

In the following, the main result of the paper, i.e., a theorem to prove the convergence of the algorithm toward a common point in a 2-dimensional space, is described. Note that, this result differs the previous contributions on gossip [15, 2] as it considers an arbitrary edge selection process $\mathfrak{e}$ where edges are chosen from a time-varying set $E(t)$.

Theorem 5 *Let us consider a gossip algorithm $\{\mathcal{S}, \mathcal{R}, \mathfrak{e}\}$, with $\mathcal{S}, \mathcal{R}$ defined respectively as in Definition (2), and Definition (3). If $\mathfrak{e}$ is such that $\forall t,\ \exists\ \Delta t\ : \mathbb{G}(t, t+\Delta t)$ is connected, then:*

$$\lim_{t\to\infty} s_{gi}(t) = R_i s_i(t) + p_i = \frac{1}{n}\sum_{i=1}^{n}\Big(R_i s_i(0) + p_i\Big), \qquad \forall\, i = 1, \ldots, n. \tag{8}$$

***Proof* 2** *By following Lemma (1), the proposed gossip algorithm can be re-written in a common global reference frame. Moreover, the state evolution can be investigated independently for each axis as follows:*

$$\begin{aligned} x(t+1) &= W(\mathfrak{e}(t))x(t), \\ y(t+1) &= W(\mathfrak{e}(t))y(t). \end{aligned}$$

Let us focus only on the $x(t)$ axis as the same holds for the $y(t)$ axis. Now, due to Lemma (2) we know that for any given interval $[t, t + \Delta t]$:

$$\hat{C}_{(t,t+\Delta t)} = \bigcap_{e_{ij}\in\mathbb{E}(t,t+\Delta t)} C(e_{ij}) = span\{\mathbf{1}_n\}.$$

In addition, due to Lemma (3) we know that for any given interval $[t, t+\Delta t]$ such that $\mathcal{G}(t+\Delta t)$ is connected the following holds:

$$\begin{aligned} &\|\Phi_{(t,t+\Delta t)}\, x - c\| < \|x - c\|, \\ &\qquad \forall\, c \in \hat{C}_{(t,t+\Delta t)}, \qquad \forall\, x \in \mathbb{R}^n \backslash \hat{C}_{(t,t+\Delta t)} \end{aligned}$$

Finally, due to Lemma (4) we know that exists a sequence of intervals $\{l_i\}$ so that:

$$d\left(x(l_i), span\{\mathbf{1}_n\}\right) \to 0.$$

Therefore, the sequence $\{x(l_i)\}$ converges in norm to some points in $span\{\mathbf{1}_n\}$, that is

$$\|x(l_i) - c\| \to 0 \qquad then \quad \{x(l_i)\} \to c, \quad c \in span\{\mathbf{1}_n\}.$$

In addition, each single matrix $W(e_{ij})$ is a symmetric row-sum matrix:

$$\mathbf{1}_n^T W(e_{ij}) = \mathbf{1}_n^T \qquad\qquad W(e_{ij})\mathbf{1}_n = \mathbf{1}_n.$$

Therefore, the sum of the vector components must be preserved over time at each iteration. This implies that for a given $c = \gamma\,\mathbf{1}_n$:

$$\sum_{i=1}^{n} c_i = \sum_{i=1}^{n} x_i(l_0), \quad \gamma = \frac{\sum_{i=1}^{n} x_i(l_0)}{n}.$$

From this it follows that:

$$x(l_i) \to \frac{\sum_{i=1}^{n} x_i(l_0)}{n}\,\mathbf{1}_n, \quad thus \quad y(l_i) \to \frac{\sum_{i=1}^{n} y_i(l_0)}{n}\,\mathbf{1}_n.$$

Therefore, for each agent i we have:

$$s_{gi}(t) \to \begin{bmatrix} \dfrac{\sum_{i=1}^{n} x_i(l_0)}{n} \\ \dfrac{\sum_{i=1}^{n} y_i(l_0)}{n} \end{bmatrix} = \frac{1}{n}\sum_{i=1}^{n}\Big(R_i s_i(0) + p_i\Big),$$

which proves the statement.

Corollary 1 *Let us consider the gossip algorithm defined by $\{\mathcal{S}, \mathcal{R}, \mathfrak{e}\}$ as in Theorem 5. If each agent initializes its state $s_i(0) = 0$ to zero, then all the agents estimates converge to the network centroid:*

$$\lim_{t\to\infty} s_{gi}(t) = R_i s_i(t) + p_i = \frac{1}{n}\sum_{i=1}^{n} p_i, \quad \forall\, i = 1, \ldots, n. \tag{9}$$

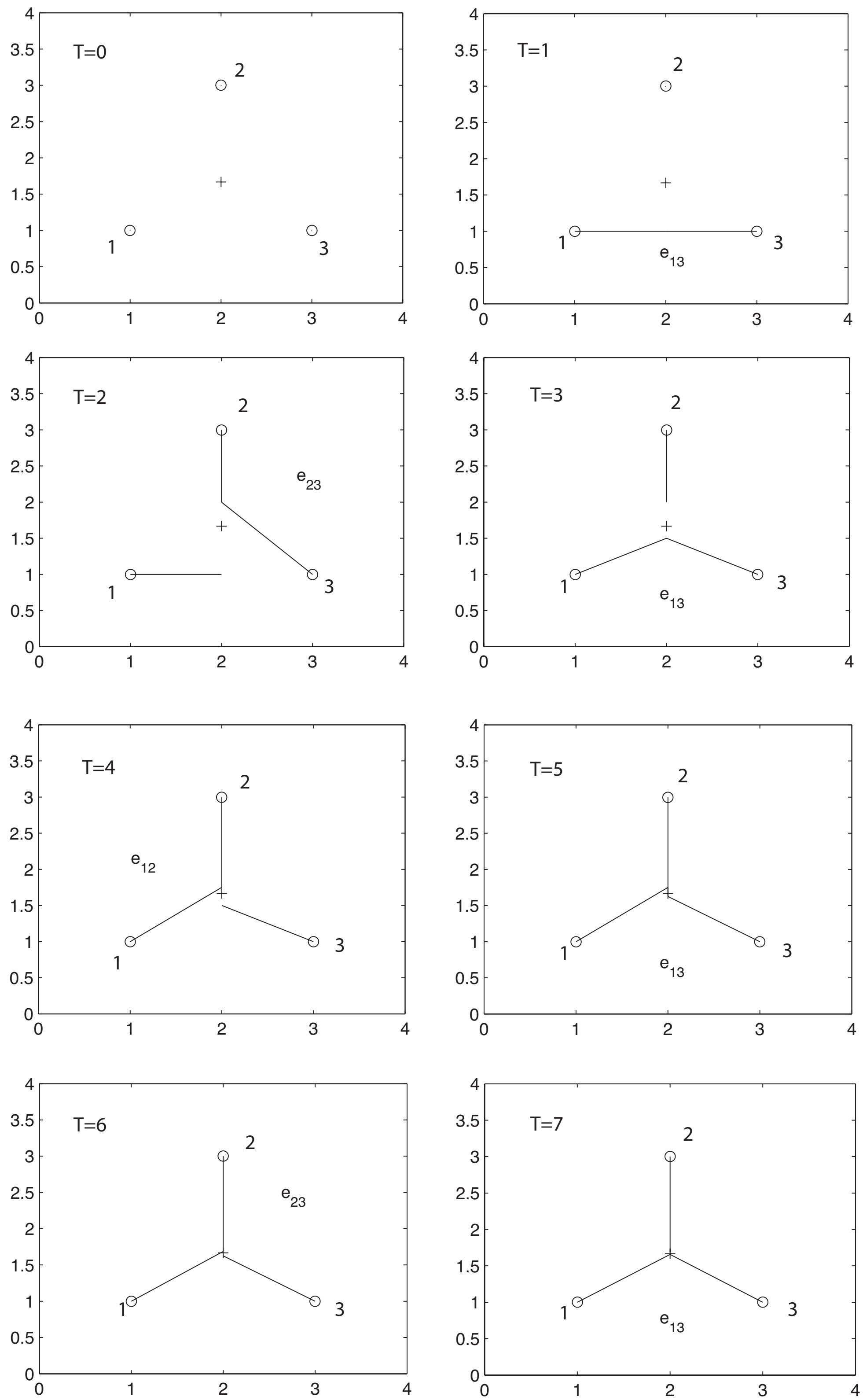

Figure 2: Example of execution of the gossip algorithm for agreement on the centroid of the network for three agents.

***Proof* 3** *The proof follows from Theorem 5 if each agent i has $s_i(0) = 0$.*

Example 1 *Fig 2 shows an example of algorithm 1 execution where the proposed interaction rule to achieve an agreement toward a common point in a decentralized fashion is used. Note that, for sake of clarity only the agreement toward the network centroid is shown. In particular, three agents labeled as $V_i = \{1, 2, 3\}$ are considered, communication links are assumed to be present between each couple of agents and their reference frames are all different but are not shown. According to Corollary 1, the algorithm is initialized at time $T = 0$ with $s_i(0) = 0$ for $i = 1, 2, 3$. At each iteration, an edge is randomly selected, and only non-trivial state updates are shown. After a sufficient number of iterations, the agents achieve a common estimate of the network centroid, each one with respect to its own reference frame.*

Algorithm 2: Reference Frame Agreement Algorithm

Data: $\mathcal{F}_i = \{f_{1,i}, f_{2,i}\}$
Result: A_i^r

- Compute the versors $r_{x,i}$ and $r_{y,i}$:

$$r_{x,i} = \frac{(f_{2,i} - f_{1,i})}{\|f_{2,i} - f_{1,i}\|} \qquad r_{y,i} = r_{x,i}^{\perp},$$

- Compute the translation vector t_i:

$$t_i = \|f_{1,i} - p_i\|,$$

- Compute the homogeneous transformation matrix A_i^r:

$$A_i^r = \begin{bmatrix} R_i^r & t_i \\ 0 & 1 \end{bmatrix}$$

---

5 Agreement on a common reference frame in a 2-D space

In Section 4 an algorithm for the agreement on a common point in a 2-D space has been described. In this section, this result is used to build an algorithm to reach an agreement on a common reference frame in a 2-D space. To this end, by exploiting Algorithm 1 according to Theorem 5, the network of agents first achieve an agreement on a set of two common points whose representation is obviously local to the agent reference frame, i.e., $\mathcal{F}_i = \{f_{1,i}, f_{2,i}\}$. Then by using Algorithm 2, each agent builds a rotation matrix A_i^r with

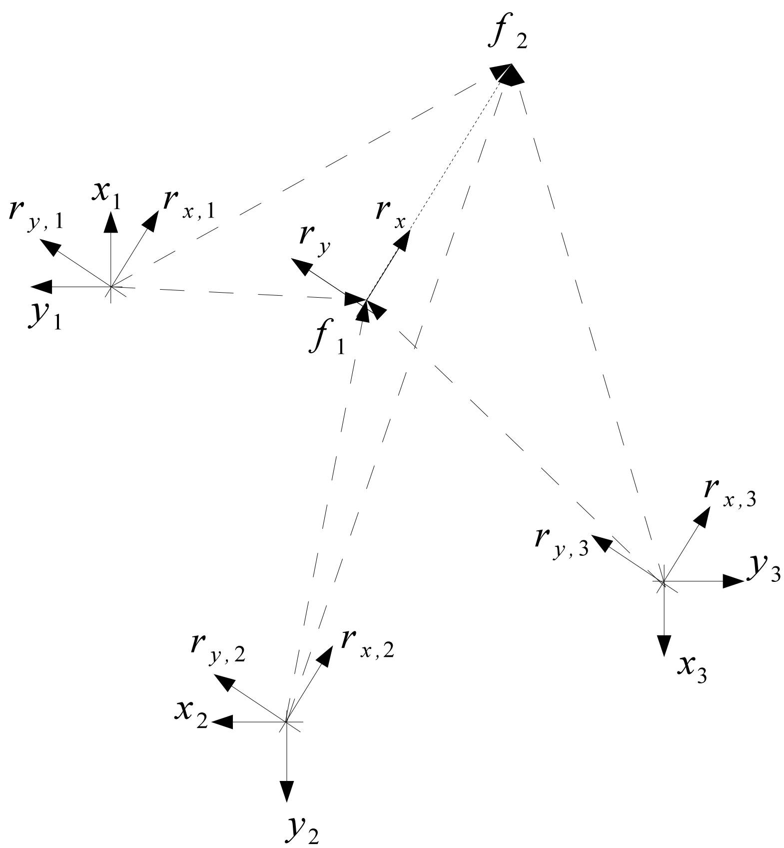

Figure 3: Example of algorithm agreement on a common reference frame for a multi-agent system composed of three agents.

respect to a common reference frame defined according to $\mathcal{F}_i$. In order to do that, the two points $\mathcal{F}_i = \{f_{1,i}, f_{2,i}\}$ can be enumerated according to the temporal order in which they have been computed. Moreover, we may have the first point $f_{1,i}$ identify the origin of the frame $O_r = f_{1,i}$ and use the second point $f_{2,i}$ to compute a common x versor, while the common y versor can be chosen to achieve a right-handed orthogonal frame. Fig. 3 shows an example of convergence toward a common reference frame for a multi-agent system composed of three agents.

Comments on measurement noise

The proposed method is inherently robust against noise in the distance measurements. This is due to the fact that since agents can communicate, they can compute the average between the distance measured at both ends. In this way, the effects of the noise on the measurements result in a symmetric contribution for both agents thus not changing the global average of their estimations. Indeed, the effect of this kind of noise consists in perturbing the local estimation while not affecting the final convergence point. The algorithm converges inside a ball around the point specified by the average of the initial measurements. Furthermore, the radius of such ball dependents on the variance and specific characteristics of the noise process. On the other hand, the proposed method is not robust against noise in the measurements with respect to the direction of the line of sight. Indeed, this contribution is not-symmetrical and not-linear, thus inaccurate direction measurements may indeed move the convergence point.

6 Conclusions

In this paper a novel approach to the problem of decentralized agreement toward a common point in space in a multi-agent system in absence of a common reference frame has been addressed. The proposed approach allows to perform an agreement on the network centroid, on a common reference frame and therefore on a common heading. Using this information a global positioning system for the agents using only local measurements can be built. Only point-to-point asynchronous communications between neighboring agents are allowed. Future work, apart from a validation on a real team of networked mobile robots, will be focused on the theoretical analysis of the algorithms robustness respect to measurement noise.

A Appendix

This section contains the proofs of some lemmas.

A.1 Proof of Lemma 2:

For any given edge e_{ij} the related matrix $W(e_{ij})$ is an orthogonal projection matrix, i.e., W is idempotent and symmetric. Moreover:

$$C(e_{ij}) = \mathrm{Ker}\,(I - W(e_{ij})) = \mathrm{Im}\,(W(e_{ij})).$$

Let us recall that for any orthogonal projection matrix $W(e_{ij})$ the following property holds [15]:

$$\left[\bigcap \mathrm{Im}\,(W(e_{ij}))\right]^{\perp} = \bigcup \mathrm{Ker}\,(W(e_{ij})).$$

Now, let us define the structure of the kernel for a generic matrix $W(e_{ij})$) as follows:

$$\mathrm{Ker}\,(W(e_{ij})) = \mathrm{span}\{[0,\,\ldots,\underbrace{1}_{i},\,\ldots 0,\,\ldots,\underbrace{-1}_{j},\,\ldots 0]^T\}.$$

Note that the vector $b_{ij} = [0,\,\ldots,\underbrace{1}_{i},\,\ldots 0,\,\ldots,\underbrace{-1}_{j},\,\ldots 0]^T$ can be thought as the column of the incidence matrix $\mathcal{I}$ [6] for the graph $\mathbb{G}(t, t+\Delta t)$ which describes the link between agent i and agent j with an arbitrary orientation chosen for convenience. At this point, by exploiting this analogy along with results coming from the graph theory we know that $\mathrm{rank}\,(\mathcal{I}) = n - c$, where n is the number of vertices and c the number of connected components. Note that, the particular choice of the orientation (if any) does not affect the number of connected components (Th. 8.3.1 [6]). Hence, if the graph $\mathbb{G}(t, t+\Delta t)$ is connected we have that:

$$\mathrm{rank}\,(\mathcal{I}) = n - 1, \quad \dim(\hat{C}_{t,t+\Delta t}) = \dim(\mathrm{Im}\,(\mathcal{I})^{\perp}) = 1.$$

Moreover, by noticing that $\mathcal{I}$ is by construction a row-sum matrix, the following holds:

$$\hat{C}_{(t,t+\Delta t)} = \mathrm{Im}\,(\mathcal{I})^{\perp} = \mathrm{span}\{\mathbf{1}_n\}.$$

$\square$

A.2 Proof of Lemma 3:

For any given edge e_{ij} the related matrix $W(e_{ij})$ is an orthogonal projection matrix, i.e., W is idempotent and symmetric. Moreover, the following holds:

$$\|W(e_{ij})\,x\| \;\le\; \|x\|, \qquad \forall\, x \in \mathbb{R}^n.$$

In our case, both the euclidian norm $\|\cdot\|$ and the infinity norm $\|\cdot\|_\infty$ are suitable. Using the euclidian norm and the fact that $c = \gamma\mathbf{1}$ with $\gamma \in \mathbb{R}$, we notice that:

$$\|x - c\|^2 = \|x\|^2 + \|c\|^2 - 2\,c^T x.$$

Therefore, the inequality (5) can be rewritten as follows:

$$\begin{aligned} \|W(e_{ij})\,x - c\| &\;\le\; \|x - c\|, \\ \|W(e_{ij})\,x\|^2 + \|c\|^2 - 2\,c^T W(e_{ij})\,x &\le \|x\|^2 + \|c\|^2 - 2\,c^T x. \end{aligned}$$

At this point by noticing that any matrix $W(e_{ij})$ is a row-sum matrix the following holds:

$$\mathbf{1}_n^T W(e_{ij}) = \mathbf{1}_n^T \qquad\qquad W(e_{ij})\mathbf{1}_n = \mathbf{1}_n.$$

Now, by recalling that for any $c \in \hat{C}_{t,t+\Delta t}$ we have $c = \gamma \cdot \mathbf{1}_n$, the following holds:

$$c^T W(e_{ij}) = \gamma \cdot \mathbf{1}_n^T\, W(e_{ij}) = \gamma \cdot \mathbf{1}_n^T = c^T.$$

An therefore the previous inequality can be rewritten as follows:

$$\begin{aligned} \|W(e_{ij})\,x\|^2 - 2\,c^T x &\;\le\; \|x\|^2 - 2\,c^T x, \\ \|W(e_{ij})\,x\|^2 &\;\le\; \|x\|^2, \end{aligned}$$

which by construction is always verified $\forall\, c \in \hat{C}_{(t,t+\Delta t)}$, and $\forall\, e_{ij} \in \mathbb{E}_{(t,t+\Delta t)}$.

A similar argument holds for inequality (9). In particular, in this case it should be noticed that for any matrix $W(e_{ij})$ the following holds:

$$\|W(e_{ij})\,x\| \;<\; \|x\|, \qquad \forall\, x \notin C(e_{ij}).$$

Therefore, if we consider the product $\Phi_{(t,t+\Delta t)} = \prod_{e_{ij} \in \mathbb{E}(t,t+\Delta t)} W(e_{ij})$, we have

$$\begin{aligned} &\|\Phi_{(t,t+\Delta t)}\, x - c\| < \|x - c\|, \\ &\qquad \forall\, c \in \hat{C}_{(t,t+\Delta t)}, \qquad \forall\, x \notin \hat{C}_{(t,t+\Delta t)} \end{aligned}$$

as since $\mathbb{G}(t, t+\Delta t)$ is connected, $\forall x \notin \hat{C}_{(t,t+\Delta t)}$ there will always be at least an edge e_{ij} so that $x \notin C(e_{ij})$ and therefore

$$\|W(e_{ij})\,x\| \;<\; \|x\|.$$

$\square$

A.3 Proof of Lemma 4

The proof is a simple consequence of the results given in Lemma (2) and Lemma (3). In particular, by exploiting Lemma (2) we have that for any sequence of intervals $\{l_i\}$:

$$\hat{C}_{(l_0,l_1)} = \hat{C}_{(l_1,l_2)} = \ldots = \hat{C}_{(l_{i-1},l_i)} = \text{span}\{\mathbf{1}_n\}.$$

Moreover, by exploiting Lemma (3) we have that for any sequence of intervals $\{l_i\}$:

$$\begin{aligned}
\|\Phi_{(l_{i-1},l_i)}\, x(l_{i-1}) - c\| &< \|x(l_{i-1}) - c\|, \\
&\vdots \\
\|\Phi_{(l_0,l_1)}\, x(l_0) - c\| &< \|x(l_0) - c\|.
\end{aligned}$$

Therefore:

$$d(x(l_i), \text{span}\{\mathbf{1}_n\}) \to 0 \qquad \forall\, c \in \text{span}\{\mathbf{1}_n\}.$$

□

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