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A Note on Isosceles Planar Graph Drawing

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ABSTRACT

We show that there exist triangulations that do not admit any planar straight-line drawing in which every face is an isosceles triangle, thus partially solving a question posed in [*Demaine, Mitchell, O'Rourke – The Open Problems Project*].

1 Introduction

Constructing planar straight-line drawings of planar graphs has always been one of the most studied topics in Graph Drawing. As every planar graph admits a planar straight-line drawing [12, 7, 10], searching for planar straight-line drawings satisfying further constraints or optimizing further aesthetic criteria constitutes the core of the topic. To provide a few examples, drawings with good vertex resolution and small area (see, e.g., [2, 9]), drawings with good angular resolution (see, e.g., [8]), drawings with few edge slopes (see, e.g., [5, 4]), drawings with sides of equal length (see, e.g., [6]), and drawings with convex faces (see, e.g., [11, 1]) have been studied.

In this note we study isosceles planar drawings of maximal planar graphs, that is, planar straight-line drawings such that all the faces are isosceles triangles. The problem of determining whether every maximal planar graph has an isosceles planar drawing has been first posed by Malkovitch during *Graph Drawing '99* and it is now part, as *Problem 69*, of the *Open Problems Project*, a well-known list of open problems in Computational Geometry and related fields maintained by Demaine, Mitchell, and O'Rourke [3].

We show that there exist maximal planar graphs that do not admit any isosceles planar drawing. In particular, the graphs considered in this note are planar 3-trees, that is a family of graphs explicitly considered in [3] when stating the problem. The result is mainly based on a simple proof that the complete graph on four vertices has no isosceles planar drawing if its outer face is drawn as a triangle with one angle larger than 120°.

The rest of the paper is organized as follows. In Sect. 2 we give some preliminaries; in Sect. 3 we present the result of this paper; in Sect. 4 we conclude with some open problems.

2 Preliminaries

A *planar straight-line drawing* of a graph is a mapping of each vertex to a distinct point of the plane and of each edge to a straight-line segment between its end-points such that no two edges intersect except, possibly, at common end-points. A planar drawing of a graph determines a circular ordering of the edges incident to each vertex. Two drawings of the same graph are *equivalent* if they determine the same circular ordering around each vertex. A *planar embedding* is an equivalence class of planar drawings. A planar drawing partitions the plane into topologically connected regions, called *faces*. The unbounded face is the *outer face*. A graph together with a planar embedding and a choice for its outer face is called *plane graph*. A plane graph is a *triangulation* when all its faces are triangles. A planar graph is *maximal* if no edge can be added to it without violating the planarity of the graph. Hence, a triangulation is a maximal planar graph together with a choice for its outer face. An *isosceles drawing* is a straight-line drawing such that all the faces are isosceles triangles. An *isosceles planar drawing* is a straight-line planar drawing such that all the faces are isosceles triangles.

3 Isosceles Planar Drawings of Triangulations

Denote by T_4 the only triangulation with 4 vertices. We have the following main lemma:

Lemma 1 *There exists no isosceles planar drawing of T_4 in which the outer face is represented by a triangle having one angle larger than or equal to* 120°.

Proof: Refer to Fig. 1. Suppose, for a contradiction, that an isosceles planar drawing Γ of T_4 exists such that the outer face is represented by a triangle Δ having one angle larger than or equal to 120°. Denote by a, b, and c the counter-clockwise order of the vertices of Δ such that $\widehat{acb} \geq 120°$. Denote by α, β, and γ the angles of Δ incident to a, b, and c, respectively. Suppose, w.l.o.g. up to a rotation of Γ, that $\overline{ab}$ is horizontal. Then, observe that c lies to the left of the vertical line through a and to the right of the vertical line through b. We prove that no point d can be placed inside $int(\Delta)$ so that triangles (a,b,d), (b,c,d), and (c,a,d) are all isosceles.

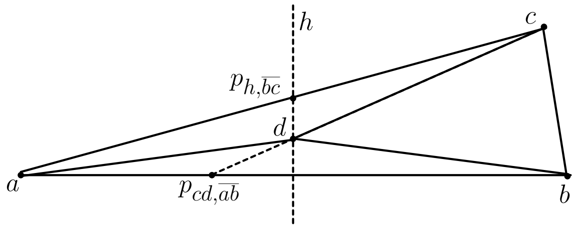

Figure 1: Illustration for the proof of Lemma 1.

First, observe that no point $p \in int(\Delta)$ is such that $|\overline{ap}| = |\overline{ab}|$ or $|\overline{bp}| = |\overline{ab}|$, as $\overline{ab}$ is the hypothenuse of Δ. It follows that, in order for triangle (a,b,d) to be isosceles, $|\overline{ad}| = |\overline{bd}|$ holds, hence d lies on the axis h of $\overline{ab}$.

Second, observe that h intersects $\overline{ac}$ or $\overline{bc}$, when considering such segments together with their end-points. Suppose that h intersects $\overline{ac}$ in a point $p_{h,\overline{ac}}$, the case in which h intersects $\overline{bc}$ being analogous. Consider any placement of d on $h \cap int(\Delta)$ and consider triangle (c,a,d). Since $|\overline{ac}| \geq |\overline{ap_{h,\overline{ac}}}|$ and since $|\overline{ap_{h,\overline{ac}}}| > |\overline{ad}|$, then $|\overline{ac}| \neq |\overline{ad}|$. Denote by $p_{cd,\overline{ab}}$ the intersection between the line through c and d and segment $\overline{ab}$. Since $|\overline{ac}| > |\overline{cp_{cd,\overline{ab}}}|$ and since $|\overline{cp_{cd,\overline{ab}}}| > |\overline{cd}|$, then $|\overline{ac}| \neq |\overline{cd}|$. It follows that, in order for triangle (c,a,d) to be isosceles, $|\overline{ad}| = |\overline{cd}|$ holds.

Third, observe that $\alpha \leq 30°$. Namely, since $|\overline{ac}| \geq |\overline{bc}|$ then $\beta \geq \alpha$; since $\gamma \geq 120°$ and $\alpha = 180° - \beta - \gamma$ then $\alpha \leq 30°$ follows.

Finally, consider triangle (b,c,d). Since $|\overline{cd}| = |\overline{ad}|$ and since $|\overline{bd}| = |\overline{ad}|$, it follows that $|\overline{cd}| = |\overline{bd}|$. Then, $\widehat{bcd} = \widehat{cbd}$. However, $\widehat{bcd} = \gamma - \widehat{acd}$; since $\widehat{acd} = \widehat{cad} < \alpha$ then $\widehat{bcd} > \gamma - \alpha$; hence $\widehat{bcd} > 90°$. Therefore, $\widehat{bcd} + \widehat{cbd} > 180°$ holds, thus providing a contradiction and proving the lemma. □

We obtain the following.

Theorem 1 *There exists an infinite class of triangulations that admit no isosceles planar drawing.*

Proof: Refer to Fig. 2. Consider the triangulation T obtained by considering a 3-cycle (a,b,c), by inserting a vertex d inside (a,b,c), by connecting d with a,b, and c, by inserting a vertex e (resp. f, resp. g) inside (a,b,d) (resp. inside (b,c,d), resp. inside (c,a,d)), and by connecting e with a,b, and d (resp. connecting f with b,c, and d, resp. connecting g with c,a, and d). In any straight-line drawing of T the angles $\widehat{adb}$, $\widehat{bdc}$, and

$\widehat{cda}$ sum up to 360°. Hence, at least one of such angles, say $\widehat{adb}$, is at least 120°. As in any planar straight-line drawing of T cycle (a, b, d) is a triangle and vertex e is inside such a triangle, Lemma 1 applies, thus implying that there is no isosceles planar drawing of T. Clearly, any triangulation containing T as a subgraph in such a way that no cycle of T encloses a vertex not in T has no isosceles planar drawing. □

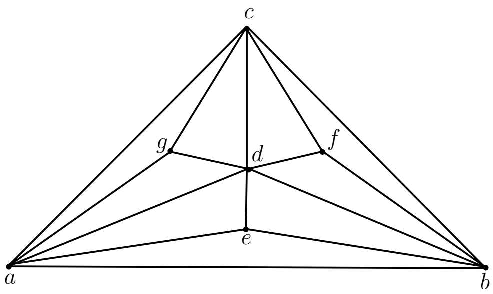

Figure 2: A triangulation T with no isosceles planar drawing.

The existence of triangulations with no isosceles planar drawing immediately implies the existence of maximal planar graphs with no isosceles planar drawing, as shown in the following.

Theorem 2 *There exists an infinite class of maximal planar graphs that admit no isosceles planar drawing.*

Proof: Consider any maximal planar graph G containing two copies T_1 and T_2 of the triangulation T, described in the proof of Theorem 1, so that no cycle of T_1 encloses a vertex not in T_1 and no cycle of T_2 encloses a vertex not in T_2. Consider any planar drawing of G. If the outer face is an internal face of T_1 (resp. of T_2), then the planar graph corresponding to the plane graph T_2 (resp. T_1) is embedded as in T_2 (resp. as in T_1). As no face is internal to both T_1 and T_2 the theorem follows from Theorem 1. □

4 Conclusions

In this note we have proved that not all the triangulations admit a planar straight-line drawing such that each face is an isosceles triangle, thus solving a problem posed in [3]. In the same problem page, two problems strictly related to the one studied in this note are mentioned.

Problem 1 *What is the time complexity of deciding whether a triangulation admits an isosceles planar drawing?*

Problem 2 *Does every triangulation admit a (possibly non-planar) isosceles drawing?*

Concerning Problem 2, it is observed in [3] that every planar 3-tree admits an isosceles non-planar drawing. Such a drawing is obtained by placing each vertex at the centroid of the face in which it has to be inserted.

References

[1] N. Chiba, T. Yamanouchi, and T. Nishizeki. Linear algorithms for convex drawings of planar graphs. In J. A. Bondy and U. S. R. Murty, editors, *Progress in Graph Theory*, pages 153–173. Academic Press, New York, NY, 1984.

[2] H. de Fraysseix, J. Pach, and R. Pollack. How to draw a planar graph on a grid. *Combinatorica*, 10(1):41–51, 1990.

[3] E. D. Demaine, J. S. B. Mitchell, and J. O'Rourke. The Open Problems Project. http://maven.smith.edu/∼orourke/TOPP/Welcome.html.

[4] V. Dujmovic, D. Eppstein, M. Suderman, and D. R. Wood. Drawings of planar graphs with few slopes and segments. *Comput. Geom.*, 38(3):194–212, 2007.

[5] V. Dujmovic, M. Suderman, and D. R. Wood. Graph drawings with few slopes. *Comput. Geom.*, 38(3):181–193, 2007.

[6] P. Eades and N. C. Wormald. Fixed edge-length graph drawing is NP-hard. *Discr. Appl. Math.*, 28(2):111–134, 1990.

[7] I. Fary. On straight line representions of planar graphs. *Acta. Sci. Math.*, 11:229–233, 1948.

[8] S. M. Malitz and A. Papakostas. On the angular resolution of planar graphs. *SIAM J. Discr. Math.*, 7(2):172–183, 1994.

[9] W. Schnyder. Embedding planar graphs on the grid. In *SODA '90*, pages 138–148, 1990.

[10] S. K. Stein. Convex maps. *Proc. Amer. Math. Soc.*, 2:464–466, 1951.

[11] W. T. Tutte. Convex representations of graphs. *Proc. Lond. Math. Soc.*, 10:304–320, 1960.

[12] K. Wagner. Bemerkungen zum vierfarbenproblem. *Jahresbericht. German. Math.-Verein*, 2:26–32, 1936.