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# Succinct Greedy Drawings May Be Unfeasible

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## ABSTRACT

A greedy drawing is a straight-line drawing that contains a distance-decreasing path for every pair of nodes. A path $(v_0, v_1, \ldots, v_m)$ is distance-decreasing if $d(v_i, v_m) < d(v_{i-1}, v_m)$, for $i = 1, \ldots, m$. Greedy drawings easily support geographic greedy routing. Hence, a natural and practical problem is the one of constructing greedy drawings in the plane, using few bits for representing vertex Cartesian coordinates and using the Euclidean distance as a metric. We show that this may be unfeasible.

# 1 Introduction

In *geographic routing* nodes forward packets based on their geographic locations, and a very simple way to do that is *greedy routing*. In greedy routing a node knows its location, the location of its neighbors, and the location of the packet's destination. Based on this information, a node forwards the packet to a neighbor that is *closer than itself* to the destination's geographic location.

Unfortunately, greedy routing has two weaknesses. First, GPS devices, typically used to determine coordinates, are expensive and increase the energy consumption of the nodes. Second, a bad interaction between network topology and location of nodes can lead to situations in which the communication fails because a *void* has been reached, i.e., a packet has reached a node whose neighbors are all farther from the destination than the node itself.

A brilliant solution to the greedy routing weaknesses has been proposed by Rao *et al.*, who in [12] proposed a protocol in which nodes are assigned *virtual coordinates* so that they apply the standard greedy routing algorithm relying on virtual locations rather than on the geographic coordinates. Clearly, virtual coordinates need not to reflect the nodes actual positions. Hence, they can be suitably chosen to guarantee that the greedy routing algorithm delivers packets.

After the publication of [12], intense research efforts have been devoted to determine: (i) Which network topologies admit a virtual coordinate assignment such that greedy routing is guaranteed to work. (ii) Which distance metrics, which system of coordinates, and how many dimensions are suitable for virtual coordinates. (iii) How many bits are needed to represent the vertex coordinates.

From a graph-theoretic point of view, Problem (i) can be stated as follows: Which are the graphs that admit a *greedy drawing*, i.e., a straight-line drawing such that, for every pair of nodes $u$ and $v$, there exists a *distance-decreasing path*? A path $(v_0, v_1, \ldots, v_m)$ is distance-decreasing if $d(v_i, v_m) < d(v_{i-1}, v_m)$, for $i = 1, \ldots, m$. This formulation of the problem gives a clear perception of how greedy routing can be seen as a "bridge" problem between theory of routing and Graph Drawing. This explains why it attracted attention in both areas.

Concerning drawings in the plane adopting the Euclidean distance, Papadimitriou and Ratajczak [10] showed that $K_{k,5k+1}$ has no greedy drawing, for $k \geq 1$. Further, they observed that, if a graph $G$ has a greedy drawing, then any graph containing $G$ as a spanning subgraph has a greedy drawing. Dhandapani [2] showed, with an existential proof based on an application of the Knaster-Kuratowski-Mazurkievicz Theorem [7] to the Schnyder methodology [13], that every *triangulation* admits a greedy drawing. Algorithms for constructing greedy drawings of triangulations and triconnected planar graphs are proposed in [1, 8]. In [8] it is also proved that there exist trees not admitting any greedy drawing.

Concerning Problem (ii), it has been shown that virtual coordinates guarantee greedy routing to work for every tree, and hence for every connected topology, when they can be chosen in the hyperbolic plane [6].

Unfortunately, the algorithms mentioned above construct greedy drawings that are not *succint*, i.e. they require $\Omega(n \log n)$ bits in the worst case for representing the vertex coordinates (Problem (iii)). This makes them unsuitable for the motivating application of greedy routing. For solving this drawback, Eppstein and Goodrich [4] proposed an elegant algorithm for greedy routing in the hyperbolic plane representing vertex coordinates with $O(\log n)$ bits. However, the perhaps most natural question of whether greedy drawings can be constructed in the plane using $O(\log n)$ bits for representing vertex Cartesian coordinates and using the Euclidean distance as a metric was, up to now, open.

This paper gives a negative answer to the above question.

**Theorem 1** *For infinitely many $n$, there exists a $(3n+3)$-node greedy-drawable tree that requires $b^n$ area in any greedy drawing in the plane, under any finite resolution rule, for some constant $b > 1$.*

We state the theorem in terms of area requirement instead of required number of bits to emphasize its interest even for the Graph Drawing field. In fact, Theorem 1 is one of the few results (e.g., [3]) showing that certain families of graph drawings require exponential area. Also, greedy drawings are a kind of *proximity drawings* [9, 5], a class of graph drawings, including Euclidean Minimum Spanning Trees, for which very few is known about the area requirement [11].

The paper is organized as follows. In Sect. 2, we introduce some definitions and preliminaries; in Sect. 3 we prove that there exists an $n$-node tree $T_n$ requiring exponential area in any greedy drawing; in Sect. 4 we show an algorithm for constructing a greedy drawing of $T_n$ on an exponential-size grid; finally, in Sect. 5 we conclude and present some open problems.

# 2 Definitions and Preliminaries

A *tree* is a connected acyclic graph. The *degree of a node* is the number of edges incident to it. A *leaf* is a node with degree 1. A *leaf edge* is an edge incident to a leaf. A *path* is a tree in which every node other than the leaves has degree 2. A *caterpillar* is a tree in which the removal of all the leaves and all the leaf edges yields a path, called *spine* of the caterpillar. In a caterpillar, nodes and edges of the spine are called *spine nodes* and *spine edges*, respectively.

A *planar drawing* of a graph is a mapping of each node to a distinct point of the plane and of each edge to a Jordan curve between its endpoints such that no two edges intersect except, possibly, at common endpoints. A *straight-line drawing* is such that all the edges are straight-line segments. A planar drawing determines a circular ordering of the edges incident to each node. Two drawings of the same graph are *equivalent* if they determine the same circular ordering around each node. An *embedding* is an equivalence class of planar drawings.

The *area* of a straight-line drawing is the area of its convex hull. The concept of area of a drawing only makes sense for a fixed *resolution rule*, i.e., a rule that does not allow, e.g., vertices to be arbitrarily close (*vertex resolution rule*), or edges to be arbitrarily short (*edge resolution rule*). In fact, without any of such rules, one could construct arbitrarily small drawings with arbitrarily small area. In the following, we derive a lower bound valid under any of such rules. Namely, we will prove that, in any greedy drawing of an $n$-node tree $T_n$, the ratio between the lengths of the longest and the shortest edge of the drawing is exponential in $n$. Such a drawing requires exponential area when a resolution rule is fixed.

We now state some basic properties of the greedy drawings of trees.

The *cell* of a node $v$ in a drawing is the set of all the points in the plane that are closer to $v$ than to any of its neighbors.

**Lemma 1** *(Papadimitriou and Ratajczak [10]) A drawing is greedy if and only if the cell of each node $v$ contains no node other than $v$.*

We remark that the cell of a leaf node $v$ is the half-plane containing $v$ and delimited by the axis of the segment having $v$ as an endpoint.

**Lemma 2** *Given a greedy drawing $\Gamma$ of a tree $T$, any subtree of $T$ is represented in $\Gamma$ by a greedy drawing.*

**Proof:** Suppose, for a contradiction, that a subtree $T'$ of $T$ exists not represented in $\Gamma$ by a greedy drawing. Then, there exists a pair of nodes $(u, v)$ such that the only path in $T'$ from $u$ to $v$ is not distance-decreasing. However, such a path is also the only path from $u$ to $v$ in $T$, a contradiction. □

**Corollary 1** *Given a greedy drawing $\Gamma$ of a tree $T$ and given any edge $(u, v)$ of $T$, the subtree $T'$ of $T$ that contains $u$ (resp. $v$) and that is obtained by removing edge $(u, v)$ from $T$ completely lies in $\Gamma$ in the half-plane containing $u$ (resp. $v$) and delimited by the axis of $(u, v)$.*

**Proof:** Suppose, for a contradiction, that there exists a node $w$ of $T'$ that lies in $\Gamma$ in the half-plane containing $v$ (resp. $u$) and delimited by the axis of $(u, v)$. Then, $d(v, w) < d(u, w)$ (resp. $d(u, w) < d(v, w)$). Consider the subtree $T''$ of $T$ that consists of $T'$ and of edge $(u, v)$. The only path from $v$ to $w$ (resp. from $u$ to $w$) in $T''$ passes through $u$ (resp. through $v$), hence it is not distance-decreasing. By Lemma 2, $\Gamma$ is not a greedy drawing of $T$, a contradiction. □

**Lemma 3** *Any greedy drawing of a tree is planar.*

**Proof:** Suppose, for a contradiction, that there exists a tree $T$ admitting a non-planar greedy drawing $\Gamma$. Let $e_1 = (u, v)$ and $e_2 = (w, z)$ be two edges that cross in $\Gamma$. Edges $e_1$ and $e_2$ are not adjacent, otherwise they would overlap and $\Gamma$ would not be greedy. Then, there exists an edge $e_3 \neq e_1, e_2$ in the only path connecting $u$ to $w$. Corollary 1 implies that $e_1$ and $e_2$ lie in distinct half-planes delimited by the axis of $e_3$, hence they do not cross, a contradiction. □

**Lemma 4** *In any greedy drawing of a tree $T$, the angle between two adjacent segments is strictly greater than $60°$.*

**Proof:** Consider any greedy drawing of $T$ in which the angle between two adjacent segments $\overline{w_1w_2}$ and $\overline{w_2w_3}$ is no more than $60°$. Then, either $|\overline{w_1w_3}| \leq |\overline{w_1w_2}|$ or $|\overline{w_1w_3}| \leq |\overline{w_2w_3}|$, say $|\overline{w_1w_3}| \leq |\overline{w_2w_3}|$. Since $d(w_1, w_3) \leq d(w_2, w_3)$, the only path $(w_1, w_2, w_3)$ from $w_1$ to $w_3$ in $T$ is not distance-decreasing. □

In the following we define a family of trees with $3n + 3$ nodes, for every $n \geq 2$, that will be exploited in order to prove Theorem 1. Refer to Fig. 1.

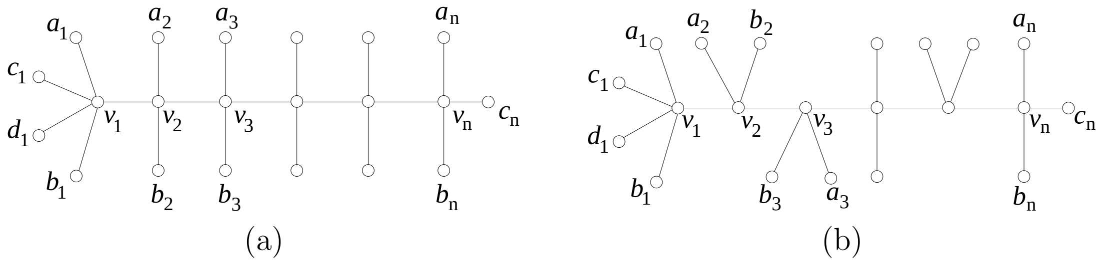



Figure 1: Two embeddings of caterpillar $T_n$. In (a) all spine nodes are central nodes. In (b) node $v_2$ is a bottom node and node $v_3$ is a top node.

**Definition 1** *Let $T_n$ be a caterpillar with spine $(v_1, v_2, \dots, v_n)$ such that $v_1$ has degree 5 and $v_i$ has degree 4, for each $i = 2, 3, \dots, n$. Let $a_1$, $b_1$, $c_1$, and $d_1$ be the leaves of $T_n$ adjacent to $v_1$, let $a_i$ and $b_i$ be the leaves of $T_n$ adjacent to $v_i$, for $i = 2, 3, \dots, n-1$, and let $a_n$, $b_n$, and $c_n$ be the leaves of $T_n$ adjacent to $v_n$.*

Distinct embeddings of $T_n$ differ for the order of the edges incident to the spine nodes. More precisely, the clockwise order of the edges incident to each node $v_i$ is one of the following: 1) $(v_{i-1}, v_i)$, then a leaf edge, then $(v_i, v_{i+1})$, then a leaf edge: $v_i$ is a *central node*; 2) $(v_{i-1}, v_i)$, then two leaf edges, then $(v_i, v_{i+1})$: $v_i$ is a *bottom node*; or 3) $(v_{i-1}, v_i)$, then $(v_i, v_{i+1})$, then two leaf edges: $v_i$ is a *top node*. Node $v_1$ is always considered as a central node.

# 3 The Lower Bound

In this section we prove that any greedy drawing of $T_n$ requires exponential area.

The proof is based on the following intuitions: (i) For any central node $v_i$ there exists a "small" convex region containing all the spine nodes $v_j$, for $j > i$, and their adjacent leaves (Lemma 5). (ii) If $v_i$ is a central node, with $i \leq n-3$, then $v_{i+1}$ must be a central node (Lemma 9). (iii) The slopes of edges $(v_i, a_i)$, $(v_i, v_{i+1})$, and $(v_i, b_i)$ incident to a central node $v_i$ must be in a certain range, which is more restricted for the edges incident to $v_{i+1}$ than for those incident to $v_i$ (Lemmata 6–7). (iv) If the angle between $(v_i, a_i)$ and $(v_i, b_i)$ is too small, then $v_j$, $a_j$, and $b_j$, with $j \geq i+2$, can not be drawn (Lemma 11). (v) The ratio between the length of the edges incident to consecutive central nodes of the spine is constant, if the angles incident to such nodes are large enough (Lemma 10).

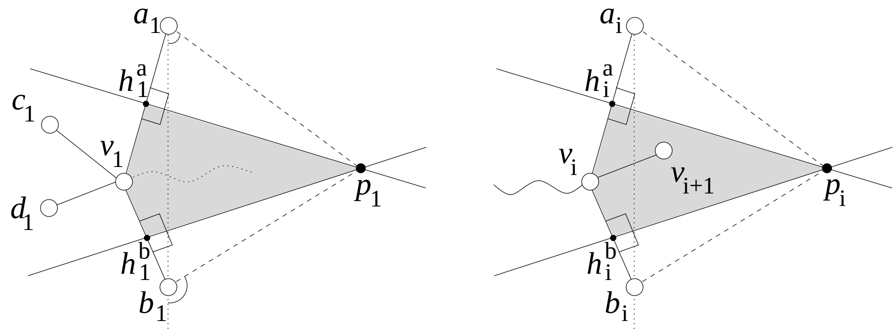
First, we discuss some properties of the slopes of the edges in the drawing. Second, we argue about the exponential decrease of the edge lengths.

## 3.1 Slopes

Consider any drawing of $v_1$ and of its adjacent leaves; rename such leaves so that their counter-clockwise order around $v_1$ is $a_1$, $c_1$, $d_1$, $b_1$, and $v_2$.

In the following, when we refer to an angle $\widehat{v_1 v_2 v_3}$, we mean the angle that brings the half-line from $v_2$ through $v_1$ to coincide with the half-line from $v_2$ through $v_3$ by a counter-clockwise rotation.

**Property 1** $\widehat{b_1 v_1 a_1} < 180°$.




(a) (b)

Figure 2: (a) Region $R_1$ contains the drawing of $T_n \setminus \{a_1, b_1, c_1, d_1, v_1\}$. The slopes of $\overline{a_1 p_1}$ and $\overline{b_1 p_1}$ are shown. (b) Region $R_i$ contains the drawing of path $(v_{i+1}, v_{i+2}, \ldots, v_n)$ and of its adjacent leaves.

**Proof:** By Lemma 4, $\widehat{a_1 v_1 c_1} > 60°$, $\widehat{c_1 v_1 d_1} > 60°$, and $\widehat{d_1 v_1 b_1} > 60°$. □

Now we argue that, for any central node $v_i$, there exists a "small" convex region that contains all the spine nodes $v_j$, for $j > i$, and their adjacent leaves.

Given a central node $v_i$, denote by $R_i$ the convex region delimited by $\overline{v_i a_i}$, by $\overline{v_i b_i}$, and by the axes of such segments (see Fig. 2.b).

**Lemma 5** *Suppose that $v_i$ is a central node. Then, $\widehat{b_i v_i a_i} < 180°$ and the drawing of path $(v_{i+1}, v_{i+2}, \ldots, v_n)$ and of its adjacent leaves lies in $R_i$.*

When $i = 1$, by Property 1 and Lemma 1, the drawing of $T_n \setminus \{a_1, b_1, c_1, d_1, v_1\}$ lies in $R_1$ (see Fig. 2.a). The proof of Lemma 5 is completed after Lemma 6.

Denote by $p_i$ the intersection between the axes of $\overline{v_i a_i}$ and $\overline{v_i b_i}$, and by $h_i^a$ ($h_i^b$) the midpoint of $\overline{v_i a_i}$ (resp. $\overline{v_i b_i}$). Assume that $x(a_i) = x(b_i)$, $x(v_i) < x(a_i)$, and $y(a_i) > y(b_i)$. Such a setting can be achieved w.l.o.g. up to a rotation/mirroring of the drawing and a renaming of the leaves. In the following, whenever a central node $v_i$ is considered, the drawing is rotated/mirrored and the leaves adjacent to $v_i$ are renamed so that $x(a_i) = x(b_i)$, $x(v_i) < x(a_i)$, and $y(a_i) > y(b_i)$. Let $slope(u, v)$ be the angle bringing the half-line from $u$ directed downward to coincide with the half-line from $u$ through $v$ by a counter-clockwise rotation (see Fig. 2.a). Further, let $slope_\perp(u, v)$ be equal to $slope(u, v) - 90°$. The following lemma holds assuming that the inductive hypothesis of Lemma 5 is verified.

**Lemma 6** *Let $(v_j, a_j)$ be a leaf edge of $T_n$, with $j > i$. Then, $slope_\perp(b_i, p_i) < slope(v_j, a_j) < slope_\perp(p_i, a_i)$.*

**Proof:** Refer to Fig. 3. By inductive hypothesis, both $v_j$ and $a_j$ lie in $R_i$. Then, $slope(v_i, b_i) < slope_\perp(b_i, p_i)$, since $slope(h_i^b, p_i) < slope(b_i, p_i)$. Analogously, $slope_\perp(p_i, a_i) < slope(v_i, a_i)$. Suppose, for a contradiction, that $slope_\perp(b_i, p_i) < slope(v_j, a_j) < slope_\perp(p_i, a_i)$ does not hold. Then, the cell of $a_j$ contains at least one out of $a_i$, $v_i$, and $b_i$. By Lemma 1, the drawing is not greedy. □

An analogous lemma holds for each leaf edge $(v_j, b_j)$, with $j > i$.

**Proof of Lemma 5:** Suppose, w.l.o.g. up to a renaming of the leaves adjacent to $v_j$, that $slope(v_j, b_j) < slope(v_j, a_j)$. From the proof of Lemma 6, $slope(v_i, b_i) <$

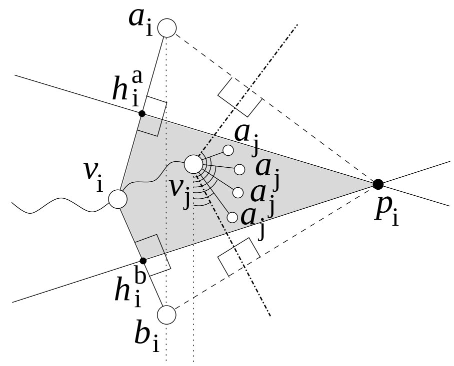



Figure 3: Possible slopes for a leaf edge $(v_j, a_j)$.

$slope_\perp(b_i, p_i)$ and $slope_\perp(p_i, a_i) < slope(v_i, a_i)$. By Lemma 6, $slope_\perp(b_i, p_i) < slope(v_j, b_j)$ and $slope(v_j, a_j) < slope_\perp(p_i, a_i)$. Then, $slope(v_i, b_i) < slope(v_j, b_j) < slope(v_j, a_j) < slope(v_i, a_i)$; hence, $\widehat{b_j v_j a_j} < \widehat{b_i v_i a_i} < 180°$.

By Lemma 3, the drawing is planar; by Lemma 1, the cells of $a_j$ and $b_j$ do not contain any node other than $a_j$ and $b_j$, respectively. Hence, if a node $u$ is in $R_j$, then no node of any subtree of $T_n$ containing $u$ and not containing $v_j$ lies outside $R_j$. Thus, $v_{j-1}$ does not lie in $R_j$ (since a subtree of $T_n$ exists containing $v_{j-1}$, $v_i$, and not containing $v_j$); since $v_j$ is a central node, then $v_{j+1}$ (and path $(v_{j+1}, v_{j+2}, \dots, v_n)$ together with its adjacent leaves) lies inside $R_j$. □

Property 2 and Lemma 7, that are presented in the following, will be used later to prove that almost all the spine nodes are central nodes.

Let $(v_i, v_{i+1}, \dots, v_k, a_k)$ be a path defined as follows (recall that $v_i$ is assumed to be a central node): For each $j = i+1, \dots, k-1$, $v_j$ is a top node, $a_k$ is a leaf, and $(v_k, a_k)$ follows $(v_{k-1}, v_k)$ in the clockwise order of the edges around $v_k$.

**Property 2** *$\widehat{v_{j+2} v_{j+1} v_j} < 180°$, for $j = i, \dots, k-2$. Further, $\widehat{a_k v_k v_{k-1}} < 180°$.*

**Proof:** By Lemma 4, $\widehat{v_j v_{j+1} b_{j+1}}$, $\widehat{b_{j+1} v_{j+1} a_{j+1}}$, and $\widehat{a_{j+1} v_{j+1} v_{j+2}}$ are strictly greater than 60°. Analogously, the three angles incident to $v_k$ and summing up to 360° together with angle $\widehat{a_k v_k v_{k-1}}$ are strictly greater than 60°. □

**Lemma 7** *$slope(v_j, v_{j+1}) < slope(v_i, a_i)$, for $j = i, i+1, \dots, k-1$. Further, $slope(v_k, a_k) < slope(v_i, a_i)$.*

**Proof:** Inequality $slope(v_k, a_k) < slope(v_i, a_i)$ directly follows from Lemma 6. We prove that $slope(v_j, v_{j+1}) < slope(v_i, a_i)$, for $j = i, i+1, \dots, k-1$. By the assumption that $(v_i, v_{i+1})$ is a central node, segment $\overline{v_i v_{i+1}}$ is between $\overline{v_i a_i}$ and $\overline{v_i b_i}$. Hence, $slope(v_i, v_{i+1}) < slope(v_i, a_i)$. Suppose, for a contradiction, that there exists a segment $\overline{v_j v_{j+1}}$ such that $slope(v_j, v_{j+1}) \geq slope(v_i, a_i)$. Let $t$ be the smallest index such that $slope(v_t, v_{t+1}) > slope(v_i, a_i)$, with $i+1 \leq t \leq k-1$.

Consider polygon $Q_t = (a_i, v_i, v_{i+1}, \dots, v_t, a_i)$. All sides of $Q_t$ are edges of $T_n$, except for $(v_t, a_i)$. Further, $Q_t$ is convex, namely $\widehat{v_{i+1} v_i a_i} < 180°$ and $\widehat{v_{j+1} v_j v_{j-1}} < 180°$, by Property 2, $\widehat{a_i v_t v_{t-1}} < 180°$, since $slope(v_{t-1}, v_t) < slope(v_i, a_i)$, and $\widehat{v_i a_i v_t} < 180°$, since it is smaller than an angle of triangle $(a_i, v_i, p_i)$. Suppose that $\overline{v_t v_{t+1}}$ lies inside $Q_t$ (see Fig. 4.a).

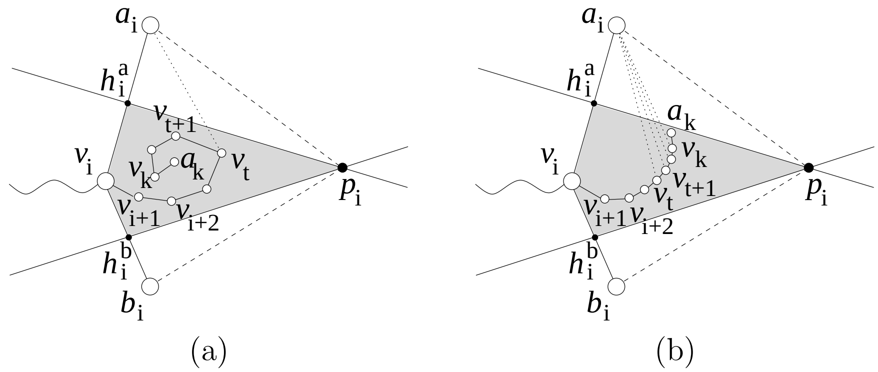



Figure 4: Illustration for the proof of Lemma 7. (a) $\overline{v_t v_{t+1}}$ lies inside $Q_t$. (b) $\overline{v_j v_{j+1}}$ lies outside $Q_j$, for each $j = t, \dots, k-1$, and $\overline{v_k a_k}$ lies outside $Q_k$.

Then, by Property 2 and by Lemma 3, $\overline{v_k a_k}$ is also inside $Q_t$. This implies that the cell of $a_k$ contains a node of $Q_t$. By Lemma 1, the drawing is not greedy. Otherwise, $\overline{v_t v_{t+1}}$ lies outside $Q_t$. By Property 2 and by $slope(v_t, v_{t+1}) > slope(v_i, a_i)$, $slope(v_{t+1}, v_{t+2}) > slope(v_i, a_i)$ holds. Then, consider polygon $Q_{t+1} = (a_i, v_i, v_{i+1}, \dots, v_t, v_{t+1}, a_i)$. Analogously as before, if $\overline{v_{t+1} v_{t+2}}$ is inside $Q_{t+1}$, then $\overline{v_k a_k}$ is also inside $Q_{t+1}$, thus contradicting the greediness of the drawing. Iterating such an argument, we obtain that $slope(v_{k-1}, v_k) > slope(v_i, a_i)$ and that $Q_k = (a_i, v_i, v_{i+1}, \dots, v_{k-1}, v_k, a_i)$ is convex (see Fig. 4.b). Finally, if $\overline{v_k a_k}$ is inside $Q_k$ the cell of $a_k$ contains a node of $Q_k$; if $\overline{v_k a_k}$ is outside $Q_k$, then, by Property 2, $slope(v_k, a_k) > slope(v_i, a_i)$, and the cell of $a_k$ contains $a_i$. By Lemma 1, the drawing is not greedy. □

Analogously, consider a path $(v_i, v_{i+1}, \dots, v_l, b_l)$ such that, for $j = i+1, \dots, l-1$, $v_j$ is a bottom node, $b_l$ is a leaf, and $(v_l, b_l)$ follows $(v_{l-1}, v_l)$ in the counter-clockwise order of the edges around $v_l$. An analogous property and an analogous lemma show that the angle between each two consecutive edges of such a path is smaller than 180° and that the slope of each edge of such a path is greater than $slope(v_i, b_i)$.

In the next lemma we give a general property for a greedy drawing of a tree $T$. Consider two edges $(u, v)$ and $(w, z)$ such that the path in $T$ from $u$ to $w$ does not contain $v$ and $z$. Suppose that $v$ and $z$ lie in the same half-plane delimited by the line through $u$ and $w$. Suppose, w.l.o.g. up to a rotation/mirroring of the drawing, that $x(u) = x(w)$, $y(u) < y(w)$, and $0° < slope(u, v), slope(w, z) < 180°$.

**Lemma 8** $slope(u, v) < slope(w, z)$.

**Proof:** Suppose, for a contradiction, that $slope(u, v) \geq slope(w, z)$. Then, either $v$ lies in the half-plane delimited by the axis of $(w, z)$ and containing $z$, or $z$ lies in the half-plane delimited by the axis of $(u, v)$ and containing $v$. Hence, by Lemma 2, the drawing is not greedy. □

## 3.2 Exponential Decreasing Edge Lengths

Now we are ready to go in the mainstream of the proof that any greedy drawing of $T_n$ requires exponential area. Such a proof is in fact based on the following three lemmata. The first of such lemmata states that a linear number of spine nodes are central nodes, in any greedy drawing of $T_n$.

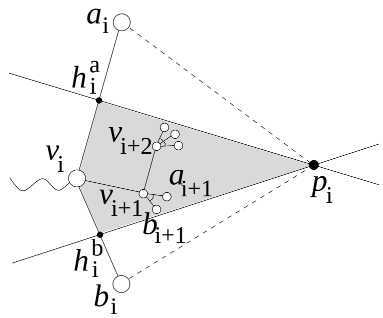



Figure 5: Illustration for the proof of Lemma 9.

**Lemma 9** *Suppose that $v_i$ is a central node, for some $i \leq n-3$. Then, $v_{i+1}$ is a central node.*

**Proof:** Refer to Fig. 5. Suppose that $v_i$ is a central node and suppose, for a contradiction, that $v_{i+1}$ is not a central node. Suppose that $v_{i+1}$ is a top node, the case in which it is a bottom node being analogous. Rename the leaves adjacent to $v_{i+1}$ in such a way that the counter-clockwise order of the neighbors of $v_{i+1}$ is $v_i$, $b_{i+1}$, $a_{i+1}$, and $v_{i+2}$. By Lemma 5, $\widehat{b_i v_i a_i} < 180°$. By Lemma 6, $slope(v_i, b_i) < slope(v_{i+1}, b_{i+1})$; by Lemma 7, $slope(v_{i+1}, v_{i+2}) < slope(v_i, a_i)$; by the hypothesis on the embedding, $slope(v_{i+1}, b_{i+1}) < slope(v_{i+1}, a_{i+1}) < slope(v_{i+1}, v_{i+2})$. Hence, $slope(v_i, b_i) < slope(v_{i+1}, b_{i+1}) < slope(v_{i+1}, a_{i+1}) < slope(v_{i+1}, v_{i+2}) < slope(v_i, a_i)$. By Lemma 4, $\widehat{b_{i+1} v_{i+1} a_{i+1}} > 60°$. It follows that $\widehat{a_{i+1} v_{i+1} v_{i+2}} < 120°$.

Suppose that $v_{i+2}$ is a central node (a top node; a bottom node). Rename the leaves adjacent to $v_{i+2}$ in such a way that the counter-clockwise order of the neighbors of $v_{i+2}$ is $v_{i+1}$, $b_{i+2}$, $v_{i+3}$, and $a_{i+2}$ (resp. $v_{i+1}$, $b_{i+2}$, $a_{i+2}$, and $v_{i+3}$; $v_{i+1}$, $v_{i+3}$, $b_{i+2}$, and $a_{i+2}$). Notice that node $v_{i+3}$ exists since $i \leq n-3$. By Lemma 8, $slope(v_{i+2}, b_{i+2}) > slope(v_{i+1}, a_{i+1})$ (resp. $slope(v_{i+2}, b_{i+2}) > slope(v_{i+1}, a_{i+1})$; $slope(v_{i+2}, v_{i+3}) > slope(v_{i+1}, a_{i+1})$). Further, by Lemma 7, $slope(v_{i+2}, a_{i+2}) < slope(v_i, a_i)$ (resp. $slope(v_{i+2}, v_{i+3}) < slope(v_i, a_i)$; $slope(v_{i+2}, a_{i+2}) < slope(v_i, a_i)$). It follows that $\widehat{b_{i+2} v_{i+2} a_{i+2}} < 120°$ (resp. $\widehat{b_{i+2} v_{i+2} v_{i+3}} < 120°$; $\widehat{v_{i+3} v_{i+2} a_{i+2}} < 120°$), hence at least one of angles $\widehat{b_{i+2} v_{i+2} v_{i+3}}$ and $\widehat{v_{i+3} v_{i+2} a_{i+2}}$ (resp. of angles $\widehat{b_{i+2} v_{i+2} a_{i+2}}$ and $\widehat{a_{i+2} v_{i+2} v_{i+3}}$; of angles $\widehat{v_{i+3} v_{i+2} b_{i+2}}$ and $\widehat{b_{i+2} v_{i+2} a_{i+2}}$) is less than 60°. By Lemma 4, the drawing is not greedy. □

The next lemma shows that, if the angles $\widehat{b_i v_i a_i}$ incident to each central node $v_i$ are large enough, then the sum of the lengths of $\overline{v_i a_i}$ and $\overline{v_i b_i}$ decreases exponentially in the number of considered central nodes.

**Lemma 10** *Let $v_i$ be a central node, with $i \leq n-3$. Suppose that both the angles $\widehat{b_i v_i a_i}$ and $\widehat{b_{i+1} v_{i+1} a_{i+1}}$ are greater than 150°. Then, the following inequality holds: $|\overline{v_{i+1} a_{i+1}}| + |\overline{v_{i+1} b_{i+1}}| \leq (|\overline{v_i a_i}| + |\overline{v_i b_i}|)/\sqrt{3}$.*

**Proof:** Refer to Fig. 6. By Lemma 9, $v_{i+1}$ is a central node. Consider the vertical line $l(v_{i+1})$ through $v_{i+1}$ and denote by $d^a_{i+1}$ and $d^b_{i+1}$ the intersection points between $l(v_{i+1})$ and the axes of $\overline{v_i a_i}$ and $\overline{v_i b_i}$, respectively. By Lemma 6, $slope_\perp(b_i, p_i) < slope(v_{i+1}, a_{i+1}), slope(v_{i+1}, b_{i+1}) < slope_\perp(p_i, a_i)$, hence $slope(v_i, b_i) < slope(v_{i+1}, a_{i+1}), slope(v_{i+1}, b_{i+1}) < slope(v_i, a_i)$. It follows that both $a_{i+1}$ and $b_{i+1}$ lie in

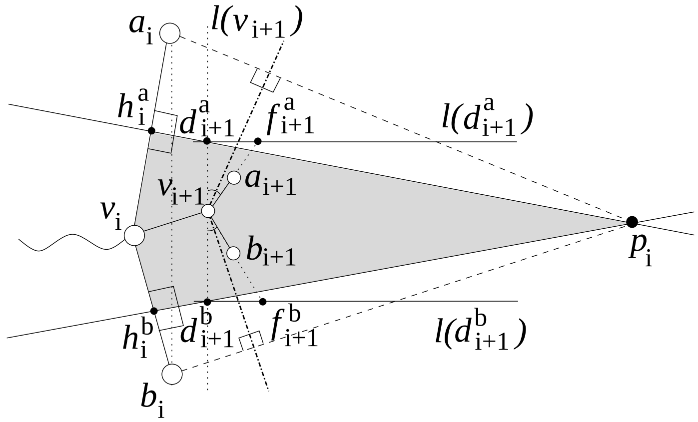



Figure 6: Illustration for the proof of Lemma 10

the half-plane delimited by $l(v_{i+1})$ and not containing $v_i$. Observe that $y(h_i^b) < y(d_{i+1}^b) < y(d_{i+1}^a) < y(h_i^a)$. Since $|\overline{h_i^b h_i^a}| < (|\overline{v_i b_i}| + |\overline{v_i a_i}|)/2$, then $|\overline{d_{i+1}^b d_{i+1}^a}| < (|\overline{v_i b_i}| + |\overline{v_i a_i}|)/2$, as well. Consider the horizontal lines $l(d_{i+1}^a)$ and $l(d_{i+1}^b)$ through $d_{i+1}^a$ and $d_{i+1}^b$, respectively. Denote by $f_{i+1}^a$ (by $f_{i+1}^b$) the intersection point between $l(d_{i+1}^a)$ and the line through $v_{i+1}$ and $a_{i+1}$ (resp. between $l(d_{i+1}^b)$ and the line through $v_{i+1}$ and $b_{i+1}$). Clearly, $|\overline{v_{i+1}a_{i+1}}| < |\overline{v_{i+1}f_{i+1}^a}|$ and $|\overline{v_{i+1}b_{i+1}}| < |\overline{v_{i+1}f_{i+1}^b}|$. Angles $\widehat{d_{i+1}^b v_{i+1} f_{i+1}^b}$ and $\widehat{f_{i+1}^a v_{i+1} d_{i+1}^a}$ are each less than 30°, namely such angles sum up to an angle which is 180° minus $\widehat{f_{i+1}^b v_{i+1} f_{i+1}^a}$, which by hypothesis is greater 150°. Hence, $|\overline{v_{i+1}a_{i+1}}| < |\overline{v_{i+1}f_{i+1}^a}| < |\overline{v_{i+1}d_{i+1}^a}|/\cos(30)$ and $|\overline{v_{i+1}b_{i+1}}| < |\overline{v_{i+1}f_{i+1}^b}| < |\overline{v_{i+1}d_{i+1}^b}|/\cos(30)$. It follows that $|\overline{v_{i+1}a_{i+1}}| + |\overline{v_{i+1}b_{i+1}}| < (|\overline{v_{i+1}d_{i+1}^a}| + |\overline{v_{i+1}d_{i+1}^b}|)/\cos(30) < 2(|\overline{v_i b_i}| + |\overline{v_i a_i}|)/2\sqrt{3}$, thus proving the lemma. □

The next lemma shows that having large angles incident to central nodes is unavoidable for almost all central nodes.

**Lemma 11** *No central node $v_i$, with $i \leq n-2$, is incident to an angle $\widehat{b_i v_i a_i}$ that is less than or equal to* 150°.

**Proof:** Refer to Fig. 7. Suppose, for a contradiction, that there exists a central node $v_i$, with $i \leq n-2$, that is incident to an angle $\widehat{b_i v_i a_i}$ that is less than or equal to 150°. Denote by $\alpha$ and $\beta$ the angles $\widehat{p_i v_i a_i}$ and $\widehat{b_i v_i p_i}$, respectively. Since triangles $(v_i, p_i, h_i^a)$ and $(a_i, p_i, h_i^a)$ are congruent, $\widehat{v_i a_i p_i} = \alpha$. Analogously, $\widehat{v_i b_i p_i} = \beta$. Summing up the angles of quadrilateral $(v_i, a_i, p_i, b_i)$, we get $\widehat{a_i p_i b_i} = 360° - 2(\alpha + \beta)$.

By Lemma 9, node $v_{i+1}$ is a central node. Consider the line through $v_{i+1}$ orthogonal to $\overline{a_i p_i}$ and denote by $g_{i+1}^a$ the intersection point between such a line and $\overline{a_i p_i}$. Further, consider the line through $v_{i+1}$ orthogonal to $\overline{b_i p_i}$ and denote by $g_{i+1}^b$ the intersection point between such a line and $\overline{b_i p_i}$. By Lemma 6, $slope_{\perp}(b_i, p_i) < slope(v_{i+1}, a_{i+1}), slope(v_{i+1}, b_{i+1}) < slope_{\perp}(p_i, a_i)$. Hence, $\widehat{b_{i+1} v_{i+1} a_{i+1}} < \widehat{g_{i+1}^b v_{i+1} g_{i+1}^a}$. Further, $\widehat{g_{i+1}^b v_{i+1} g_{i+1}^a} = 2\alpha + 2\beta - 180°$, as can be derived by considering quadrilateral $(g_{i+1}^b, v_{i+1}, g_{i+1}^a, p_i)$, that has two angles equal to 90° (the ones incident to $g_{i+1}^a$ and $g_{i+1}^b$) and one angle equal to $360° - 2(\alpha + \beta)$ (the one incident to $p_i$). Since $\alpha + \beta \leq 150°$ by hypothesis, then $\widehat{b_{i+1} v_{i+1} a_{i+1}} < \widehat{g_{i+1}^b v_{i+1} g_{i+1}^a} = 2\alpha + 2\beta - 180° \leq 120°$. However, since $v_{i+1}$ is a central node, edge $(v_{i+1} v_{i+2})$, that exists

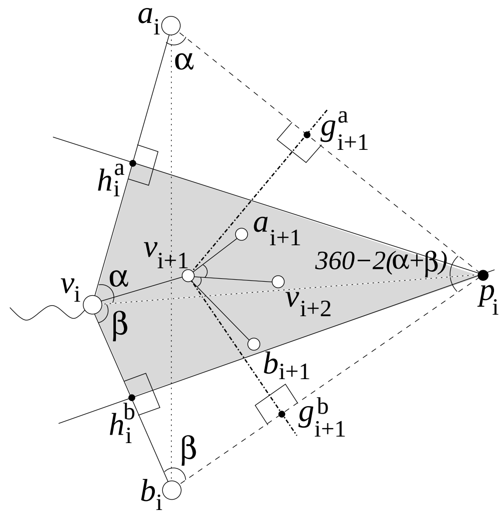



Figure 7: Illustration for the proof of Lemma 11

since $i \leq n-2$, cuts angle $\widehat{b_{i+1}v_{i+1}a_{i+1}}$. It follows that at least one of angles $\widehat{b_{i+1}v_{i+1}v_{i+2}}$ and $\widehat{v_{i+2}v_{i+1}a_{i+1}}$ is less than 60°. By Lemma 4, the drawing is not greedy. □

The previous lemmata immediately imply an exponential lower bound between the ratio of the lengths of the longest and the shortest edge of the drawing. Namely, node $v_1$ is a central node. By Lemma 9, node $v_i$ is a central node, for $i = 1, 2, \ldots, n-3$. By Lemma 11, angle $\widehat{b_i v_i a_i}$ is greater than 150°, for $i \leq n-3$. Hence, by Lemma 10, $|\overline{v_{i+1}a_{i+1}}| + |\overline{v_{i+1}b_{i+1}}| \leq (|\overline{v_i a_i}| + |\overline{v_i b_i}|)/\sqrt{3}$, for $i \leq n-4$; it follows that $|\overline{v_{n-3}a_{n-3}}| + |\overline{v_{n-3}b_{n-3}}| \leq (|\overline{v_1 a_1}| + |\overline{v_1 b_1}|)/(\sqrt{3})^{n-4}$. Since one out of $\overline{v_1 a_1}$ and $\overline{v_1 b_1}$, say $\overline{v_1 a_1}$, has length at least half of $|\overline{v_1 a_1}| + |\overline{v_1 b_1}|$, and since one out of $\overline{v_{n-3}a_{n-3}}$ and $\overline{v_{n-3}b_{n-3}}$, say $\overline{v_{n-3}a_{n-3}}$, has length at most half of $|\overline{v_{n-3}a_{n-3}}| + |\overline{v_{n-3}b_{n-3}}|$, then $|\overline{v_1 a_1}|/|\overline{v_{n-3}a_{n-3}}| \geq \frac{1}{9}(\sqrt{3})^n$, thus implying the claimed lower bound.

# 4 Drawability of $T_n$

In Section 3 we have shown that any greedy drawing of $T_n$ requires exponential area. Since in [10, 8] it has been shown that there exist trees that do not admit any greedy drawing, one might ask whether the lower bound refers to a greedy-drawable tree or not. Of course, if $T_n$ were not drawable, then the lower bound would not make sense. In this section we actually show that $T_n$ admits a greedy drawing, providing an algorithm to construct such a drawing, using a supporting exponential-size grid.

Since our algorithm draws the spine nodes in the order they appear on the spine with the degree-5 node as the last node, we revert the indices of the nodes with respect to Sects. 2 and 3, that is, node $v_i$ of $T_n$ is now node $v_{n-i+1}$.

Our algorithm constructs a drawing of $T_n$ in which all the spine nodes $v_i$ are central nodes lying on the horizontal line $y = 0$. Since each leaf node $a_i$ is drawn above line $y = 0$ and $b_i$ is placed on the symmetrical point of $a_i$ with respect to such a line, we only describe, for each $i = 1, \ldots, n$, how to draw $v_i$ and $a_i$.

In order to deal with drawings that lie on a grid, in this section we denote by $\Delta_y/\Delta_x$ the *slope* of a line (of a segment), meaning that whenever there is a horizontal distance

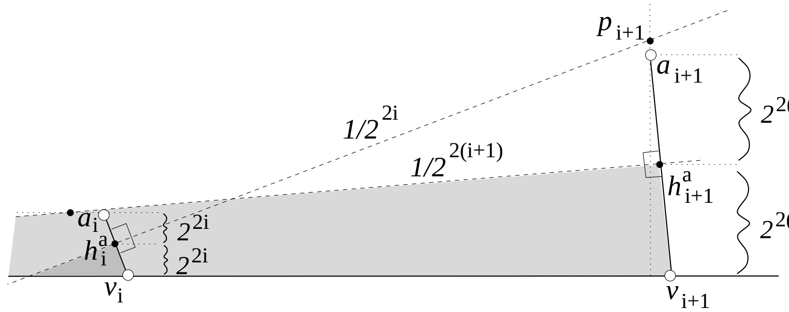



Figure 8: Illustration for the algorithm to construct a greedy drawing of $T_n$.

$\Delta_x$ between two nodes of such a line (of such a segment), then their vertical distance is $\Delta_y$.

The drawing is constructed by means of an inductive algorithm. At step $i$ of the algorithm suppose, by inductive hypothesis, that: (i) The drawing of path $(v_1, v_2, \dots, v_i)$ with its leaf nodes $a_1, a_2, \dots, a_i$ is greedy, and (ii) $y(v_i) = 0$, $y(h_i^a) = 2^{2i}$, $y(a_i) = 2^{2i+1}$, and $x(v_i) - x(h_i^a) = x(h_i^a) - x(a_i) = 1$. In the base case, place $v_1$ at $(0,0)$, $h_1^a$ at $(-1,2)$, $a_1$ at $(-2,4)$, and $c_1$ at $(-9/2,0)$.

From the above inductive hypothesis it follows that the slope of segment $\overline{v_i a_i}$ is $-2^{2i}/1$ and the slope of its axis is $1/2^{2i}$. We show step $i+1$ of the algorithm.

Place $v_{i+1}$ at point $(x(v_i)+2^{4i+3}-2, 0)$, $h_{i+1}^a$ at point $(x(v_i)+2^{4i+3}-3, 2^{2i+2})$, and $a_{i+1}$ at point $(x(v_i) + 2^{4i+3} - 4, 2^{2i+3})$. Such placements guarantee that the second hypothesis is verified. The slope of segment $\overline{v_{i+1}a_{i+1}}$ is $-2^{2(i+1)}/1$. Hence, the slope of its axis is $1/2^{2(i+1)}$. Such an axis passes through point $q_i \equiv (x(v_i) - 3, 2^{2i+1})$. Since $0 < 1/2^{2(i+1)} < 1/2^{2i}$, it follows that path $(v_1, v_2, \dots, v_i)$ with nodes $a_1, a_2, \dots, a_i$ lie below the axis of $\overline{v_{i+1}a_{i+1}}$. Finally, the axis of $\overline{v_i a_i}$ passes through point $p_{i+1} \equiv (x(v_i) + 2^{4i+3} - 4, 2^{2i} + 2^{2i+3} - 3/2^{2i})$. However, $y(p_{i+1}) > y(a_{i+1})$, since $2^{2i} + 2^{2i+3} - 3/2^{2i} > 2^{2i+3}$ as long as $2^{4i} > 3$, which holds for $i \geq 1$. This implies that the first hypothesis is verified.

When the algorithm has drawn $v_n$ and $a_n$ (and symmetrically $b_n$), $c_n$ and $d_n$ still have to be drawn. However, this can be easily done by assigning to segments $\overline{v_n c_n}$ and $\overline{v_n d_n}$ the same length as segment $\overline{v_n a_n}$ and by placing them in such a way that the angle $\widehat{b_n v_n a_n}$, which is strictly greater than 180°, is split into three angles strictly greater than 60°.

We remark that $c_n$ and $d_n$ are not placed at points with rational coordinates. However, they still obey to any resolution rule, namely they are at a distance from any node or edge of the drawing that is exponential with respect to the grid unit. Placing such nodes at grid points is possible after a scaling of the whole drawing and some non-trivial calculations. However, we preferred not to deal with such an issue since we just needed to prove that a greedy drawing exists.

# 5 Conclusions

We have shown that constructing succinct greedy drawings in the plane, when the Euclidean distance is adopted, may be unfeasible even for simple classes of trees. In fact, we proved that such trees require exponential area in any greedy drawing, under any finite resolution rule. The proof uses a mixed geometric-topological technique that allows to analyze the combinatorial space of the possible embeddings and to identify nice invariants on the slopes of the edges.

Many problems remain open in this area. By the results of Leighton and Moitra [8], every triconnected planar graph admits a greedy drawing.

*Which are the area requirements of greedy drawings of triconnected planar graphs?*

While every triconnected planar graph admits a greedy drawing, not all biconnected planar graphs and not all trees admit a greedy drawing. For example, in [8] it is shown that a complete binary tree with 31 nodes does not admit any greedy drawing. Hence, the following problem is worth studying:

*Characterize the class of trees (resp. of biconnected planar graphs) that admit a greedy drawing.*

We have shown, in Lemma 3, that every greedy drawing of a tree is planar. It would be interesting to understand whether trees are the only class of planar graphs with such a property.

*Characterize the class of planar graphs such that every greedy drawing is planar.*

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