Università degli Studi di Roma Tre
Dipartimento di Informatica e Automazione
Via della Vasca Navale, 79 – 00146 Roma, Italy

Straight-Line Rectangular Drawings of Clustered Graphs

Patrizio Angelini[†], Fabrizio Frati[†], and Michael Kaufmann[♯]

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[†]Dipartimento di Informatica e Automazione, Università Roma Tre, Italy
{angelini,frati}@dia.uniroma3.it
♯Wilhelm-Schickard-Institut für Informatik, Universität Tübingen, Germany
mk@informatik.uni-tuebingen.de

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ABSTRACT

We show that every c-planar clustered graph has a straight-line c-planar drawing in which each cluster is represented by an axis-parallel rectangle, thus solving a problem posed by Eades, Feng, Lin, and Nagamochi [*Algorithmica, 2006*].

1 Introduction

A *clustered graph* is a pair (G, T), where G is a graph, called *underlying graph*, and T is a rooted tree, called *inclusion tree*, such that the leaves of T are the vertices of G. Each internal node ν of T corresponds to the subset of vertices of G, called *cluster*, that are the leaves of the subtree of T rooted at ν.

Clustered graphs are widely used in applications where it is needed at the same time to represent relationships between entities and to group entities with semantic affinities. For example, in the Internet network, links among routers give rise to a graph; geographically close routers are grouped into areas, which in turn are grouped into Autonomous Systems.

Visualizing clustered graphs turns out to be a difficult problem, due to the simultaneous need for a readable drawing of the underlying structure and for a good rendering of the recursive clustering relationship. As for the visualization of graphs, the most important aesthetic criterion for a drawing of a clustered graph to be "nice" is commonly regarded to be the *planarity*, which however needs a refinement in order to take into account the clustering structure.

A *drawing* Γ of a clustered graph $C(G, T)$ consists of a drawing of G (each vertex is a point in the plane and each edge is a Jordan curve between its endvertices) and of a representation of each node μ of T as a simple closed region containing all and only the vertices that belong to μ. In the following, when we say "cluster", we refer both to a set of vertices and to the region representing the cluster in a drawing, the meaning of the word being clear from the context. A drawing Γ has an *edge crossing* if two edges of G cross; Γ has an *edge-region crossing* if an edge crosses the border of a cluster more than once; Γ has a *region-region crossing* if the borders of two clusters cross. A drawing is *c-planar* if it does not have edge crossings, edge-region crossings, and region-region crossings. A graph is c-planar if it has a c-planar drawing.

Given a clustered graph, testing whether it admits a c-planar drawing is a problem of unknown complexity, perhaps the most studied problem in the Graph Drawing community during the last ten years (see, e.g., [13, 11, 5, 15, 14, 4, 3, 1, 17, 2, 16]).

Suppose that a c-planar clustered graph C is given together with a c-planar embedding, that is, together with an equivalence class of c-planar drawings of C, where two c-planar drawings are equivalent if they have the same order of the edges incident to each vertex and the same order of the edges incident to each cluster. How can the graph be drawn? Such a problem has been intensively studied in the literature and a number of papers have been presented for constructing c-planar drawings of clustered graphs within many drawing conventions.

Eades *et al.* show in [8] how to construct $O(n^2)$-area c-planar orthogonal and poly-line drawings of c-planar clustered graphs with clusters drawn as axis-parallel rectangles. Di Battista *et al.* [6] show algorithms and bounds for constructing small-area drawings of c-planar clustered trees within several drawing styles. The strongest result in the area is perhaps the one that Eades *et al.* present in [7]. Namely, the authors show an algorithm for constructing c-planar straight-line drawings of c-planar clustered graphs in which each cluster is drawn as a convex region (see also a paper of Nagamochi and Kuroya [19]). Such an algorithm requires, in general, exponential area. However, in [12] Feng *et al.* have shown that such a bound is asymptotically optimal in the worst case.

In this paper we address a problem posed by Eades *et al.* in [9, 11, 7]: Does every clustered graph admit a *straight-line rectangular drawing*, i.e., a c-planar straight-line

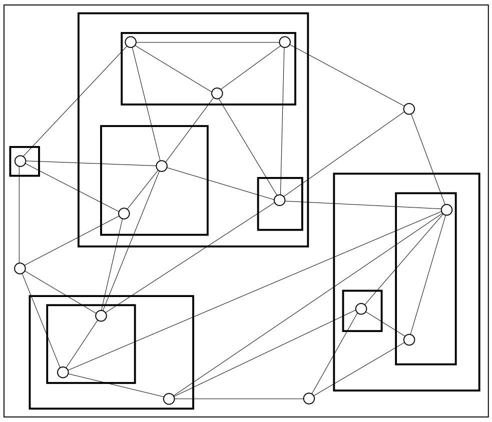


Figure 1: A straight-line rectangular drawing of a c-planar clustered graph.

drawing in which each cluster is drawn as an axis-parallel rectangle (see Fig. 1)? Eades *et al.* observe how pleasant and readable straight-line rectangular drawings are; however, they provide evidence that their algorithm [7] for constructing c-planar straight-line convex drawings of clustered graphs cannot be modified to obtain straight-line rectangular drawings without introducing edge-region crossings.

We show that every c-planar clustered graph has a straight-line rectangular drawing by means of an inductive drawing algorithm relying on a strong hypothesis, namely that a straight-line rectangular drawing of a c-planar clustered graph exists for an arbitrary *convex-separated* drawing of its outer face, that is, a drawing satisfying some properties of convexity and of visibility among vertices and clusters. The algorithm consists of three simple inductive cases, reminiscent of Fary's drawing algorithm for planar graphs [10]. When none of the three cases applies, the clustered graph is such that every cluster contains a vertex incident to the outer face of the underlying graph. We call *outerclustered graph* a clustered graph satisfying this property. We show that every outerclustered graph admits a straight-line rectangular drawing for an arbitrary convex-separated drawing of its outer face. This is done by means of an algorithm that splits an outerclustered graph into several *linearly-ordered outerclustered graphs*, i.e., outerclustered graphs in which all the vertices of the underlying graph belong to a path in the inclusion tree. Finally, a drawing algorithm is provided for constructing a straight-line rectangular drawing of any linearly-ordered outerclustered graph $C(G, T)$ for an arbitrary convex-separated drawing of its outer face. Such an inductive algorithm constructs a subgraph of G (a path plus an edge) that splits G into smaller linearly-ordered outerclustered graphs and draws such a subgraph so that the outer faces of the smaller linearly-ordered outerclustered graphs are convex-separated, thus allowing the induction to go through.

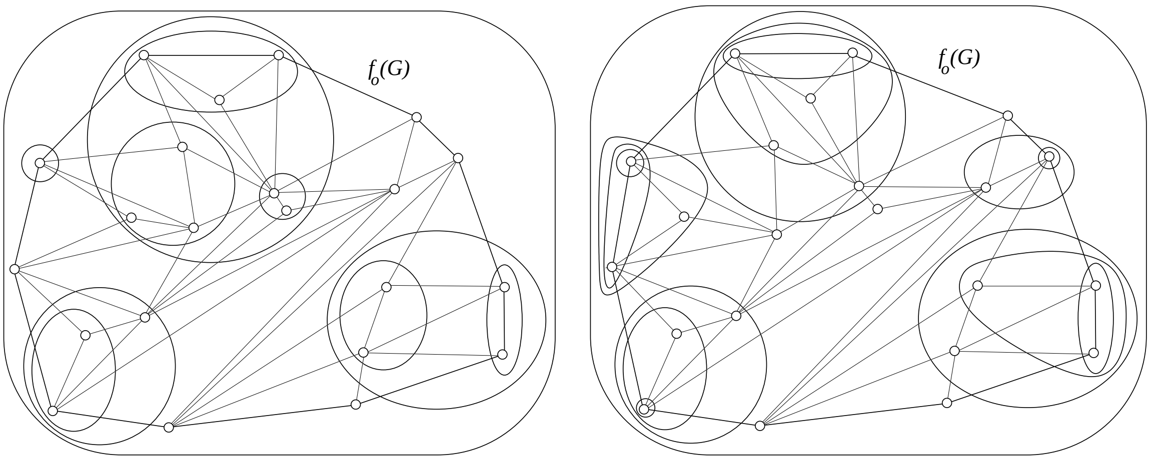
The rest of the paper is organized as follows. In Sect. 2, we introduce some preliminaries and definitions; in Sects. 3, 4, and 5, we show drawing algorithms for linearly-ordered outerclustered graphs, for outerclustered graphs, and for general clustered graphs, respectively; finally, in Sect. 6 we conclude and present some open problems.

2 Preliminaries

Let $C(G,T)$ be a clustered graph. An edge (u,v) of G is *incident to a cluster* μ *of* T if u belongs to μ and v does not. Let $\sigma(u_1, u_2, \ldots, u_k)$ be the *smallest cluster* of T containing vertices $u_1, u_2, \ldots, u_k$ of G, i.e., the node of T containing $u_1, u_2, \ldots, u_k$ and such that none of its children in T, if any, contains all of $u_1, u_2, \ldots, u_k$. A cluster is *minimal* if it contains no other clusters. A cluster μ is an ancestor (descendant) of a cluster ν if μ is an ancestor (descendant) of ν in T. C is *c-connected* if each cluster induces a connected subgraph of G.

In this paper we are interested in *straight-line rectangular drawings* of clustered graphs, i.e., c-planar drawings such that each edge is represented by a straight-line segment and each cluster is represented by an axis-parallel rectangle. From now on, "clustered graph" will always mean c-planar clustered graph, and "drawing" will always mean straight-line rectangular drawing.

A clustered graph $C(G,T)$ is *maximal* if G is a maximal planar graph. In order to prove that every clustered graph admits a straight-line rectangular drawing, it suffices to consider maximal clustered graphs. Namely, every non-maximal c-planar clustered graph $C(G,T)$ can be augmented to a maximal c-planar clustered graph by adding dummy edges to G [18]. Feng *et al.* [13] proved that a clustered graph $C(G,T)$ is c-planar if and only if edges can be added to G so that the resulting clustered graph $C'(G',T)$ is c-planar and c-connected. Since no edge can be added to a maximal c-planar clustered graph without loosing c-planarity, every maximal c-planar clustered graph is c-connected. From now on, we assume that the embedding (that is, the order of the edges incident to each vertex) and the outer face of any graph G is fixed in advance (that is, G is a *plane graph*). We denote by $f_o(G)$ the outer face of G. A clustered graph $C(G,T)$ is *internally-triangulated* if every internal face of G is delimited by a 3-cycle. Fig. 2.a shows a biconnected internally-triangulated clustered graph.



(a) (b)

Figure 2: (a) A biconnected internally-triangulated clustered graph. (b) A biconnected internally-triangulated outerclustered graph.

Let $C(G,T)$ be a clustered graph and let f be any face of G. Denote by $C_f(G_f, T_f)$ the clustered graph such that G_f is the cycle delimiting f, and such that T_f is obtained from T by removing the clusters not containing any vertex incident to f. The *outer face* *of* $C(G,T)$ is the clustered graph $C_{f_o(G)}(G_{f_o(G)}, T_{f_o(G)})$, that is simply denoted by C_{f_o}. In Sects. 3, 4, and 5, we prove that a drawing of a clustered graph can be constructed for an arbitrary drawing of its outer face satisfying some geometric properties to be described below. Then, a straight-line rectangular drawing $\Gamma(C)$ of C *completes* a straight-line rectangular drawing $\Gamma(C_{f_o})$ of C_{f_o} if the part of $\Gamma(C)$ representing C_{f_o} coincides with $\Gamma(C_{f_o})$.

We now introduce and study the following class of clustered graphs.

Definition 1 *A c-planar clustered graph $C(G,T)$ is an outerclustered graph if:*

- *O1: every cluster contains at least one of the vertices incident to the outer face $f_o(G)$ of G;*
- *O2: the border of every cluster that does not contain all the vertices incident to $f_o(G)$ intersects $f_o(G)$ exactly twice; and*
- *O3: every edge (u,v) with $\sigma(u) = \sigma(v)$ is incident to $f_o(G)$.*

Fig. 2.b shows a biconnected internally-triangulated outerclustered graph.

Let $C(G,T)$ be a biconnected internally-triangulated outerclustered graph and let $\mathcal{C}$ be any simple cycle in G such that the border of every cluster in T containing vertices of $\mathcal{C}$ and not containing all the vertices of $\mathcal{C}$ intersects $\mathcal{C}$ exactly twice. Let $C_1(G_1, T_1)$ be the clustered graph such that G_1 is the subgraph of G induced by the vertices incident to and internal to $\mathcal{C}$, and such that T_1 is the subtree of T induced by the clusters containing vertices of G_1.

Lemma 1 *$C_1(G_1, T_1)$ is a biconnected internally-triangulated outerclustered graph.*

Proof: Since G is biconnected and internally-triangulated, G_1 is biconnected and internally-triangulated, as well. We prove that C_1 satisfies Property $O1$ of Definition 1. Suppose that there exists a cluster μ in T_1 that does not contain any vertex incident to $f_o(G_1)$. Then, μ contains a vertex internal to G_1 (otherwise it would not be a cluster in T_1). Also, μ contains a vertex incident to $f_o(G)$ (otherwise C would not be an outerclustered graph). Since the border of μ is a closed curve containing a vertex inside $\mathcal{C}$ and a vertex outside $\mathcal{C}$, then either μ contains a vertex of $\mathcal{C}$ or it intersects twice the same edge of $\mathcal{C}$, in both cases contradicting the c-planarity of C. Clustered graph C_1 satisfies Property $O2$ by hypothesis. We prove that C_1 satisfies Property $O3$. Suppose that there exists an edge (u_1, v_1) such that $\sigma(u_1) = \sigma(v_1)$ and u_1 is an internal vertex of G_1. Then, u_1 is an internal vertex of G, as well, and C is not an outerclustered graph, a contradiction. □

An interesting subclass of outerclustered graphs is considered in the following.

Definition 2 *An internally-triangulated biconnected outerclustered graph $C(G,T)$ is linearly-ordered if there exists a sequence $\mu_1, \mu_2, \ldots, \mu_k$ of clusters in T and an index $1 \leq h \leq k$, such that:*

- *LO1: for each vertex v_j of G, $\sigma(v_j) = \mu_i$, for some $1 \leq i \leq k$;*
- *LO2: let v_i and v_j be any two vertices incident to $f_o(G)$ such that $\sigma(v_i) = \mu_1$ and $\sigma(v_j) = \mu_k$; then $f_o(G)$ is delimited by two monotone paths $\mathcal{P}_1 = (v_i, v_{i+1}, \ldots, v_{j-1}, v_j)$ and $\mathcal{P}_2 = (v_i, v_{i-1}, \ldots, v_{j+1}, v_j)$, i.e., paths such that, if $\sigma(v_t) = \mu_a$ and $\sigma(v_{t+1}) = \mu_b$, then $a \leq b$ if $(v_t, v_{t+1}) \in \mathcal{P}_1$ and $b \leq a$ if $(v_t, v_{t+1}) \in \mathcal{P}_2$; and*

- *LO3:* μ_{i+1} *is the parent of* μ_i*, for each* $1 \leq i < h$*, and* μ_{i+1} *is a child of* μ_i*, for each* $h \leq i < k$*.*

Fig. 3 shows a linearly-ordered outerclustered graph.

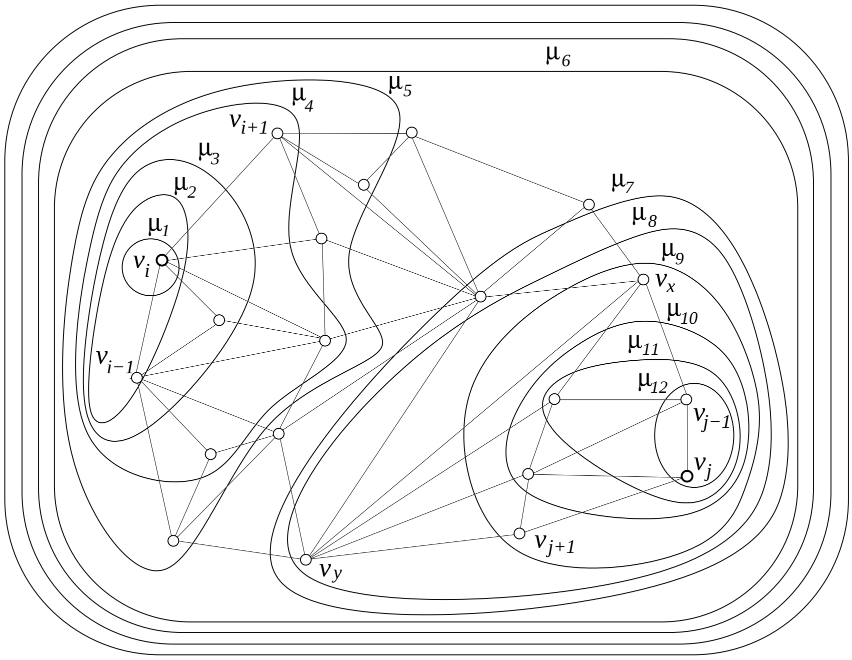


Figure 3: A linearly-ordered outerclustered graph.

In Section 3 we prove that a drawing of any linearly-ordered outerclustered graph $C(G,T)$ can be obtained even if the drawing of C_{f_o} is arbitrarily fixed. However, we deal with a constrained version of straight-line rectangular drawings defined below. By Property O2, the border of each cluster μ of T not containing all the vertices of $f_o(G)$ intersects exactly two edges $e_1(\mu)$ and $e_2(\mu)$ of $f_o(G)$.

Definition 3 *A straight-line rectangular drawing* $\Gamma(C_{f_o})$ *of* C_{f_o} *is a convex-separated drawing if:*

- *CS1: the polygon* P *representing* $f_o(G)$ *is convex;*

- *CS2: there exist two vertices* v_i *and* v_j *such that* $\sigma(v_i) = \mu_1$ *and* $\sigma(v_j) = \mu_k$*, and such that the angle of* P *incident to* v_i *and the angle of* P *incident to* v_j *are strictly less than* $180°$*; and*

- *CS3: for every pair of clusters* μ *and* ν *such that* μ *is the parent of* ν *in* T *and such that* μ *is not an ancestor of the smallest cluster containing all the vertices of* $f_o(G)$*, there exists a convex region* $R(\mu,\nu)$ *such that: (i)* $R(\mu,\nu)$ *is entirely contained inside* $\mu \cap (P \cup int(P))$*; (ii) for any cluster* $\mu' \neq \mu$ *and any child* ν' *of* μ'*,* $R(\mu,\nu)$ *does not intersect neither* $R(\mu',\nu')$ *nor the border of* μ'*; (iii)* $R(\mu,\nu) \cap P$ *consists of two continuous lines* $l_1(\mu,\nu)$ *and* $l_2(\mu,\nu)$ *such that* $l_1(\mu,\nu)$ *belongs to the line segment representing* $\mathcal{P}_1$ *in* $\Gamma(C_{f_o})$ *and* $l_2(\mu,\nu)$ *belongs to the line segment representing* $\mathcal{P}_2$ *in* $\Gamma(C_{f_o})$*; further, at least one endpoint of* $l_1(\mu,\nu)$ *(resp. of* $l_2(\mu,\nu)$*) lies on* $e_1(\nu)$ *(resp. on* $e_2(\nu)$*).*

Fig. 4 shows a convex-separated drawing of the outer face of a linearly-ordered outer-clustered graph.

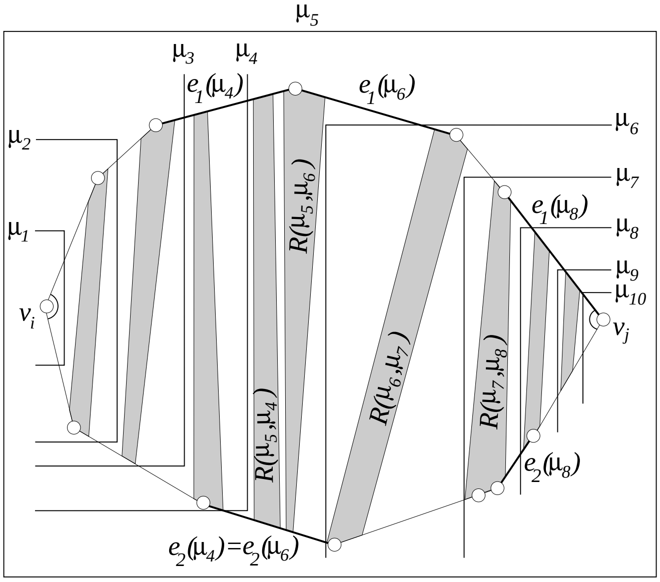


Figure 4: A convex-separated drawing of the outer face of a linearly-ordered outerclustered graph.

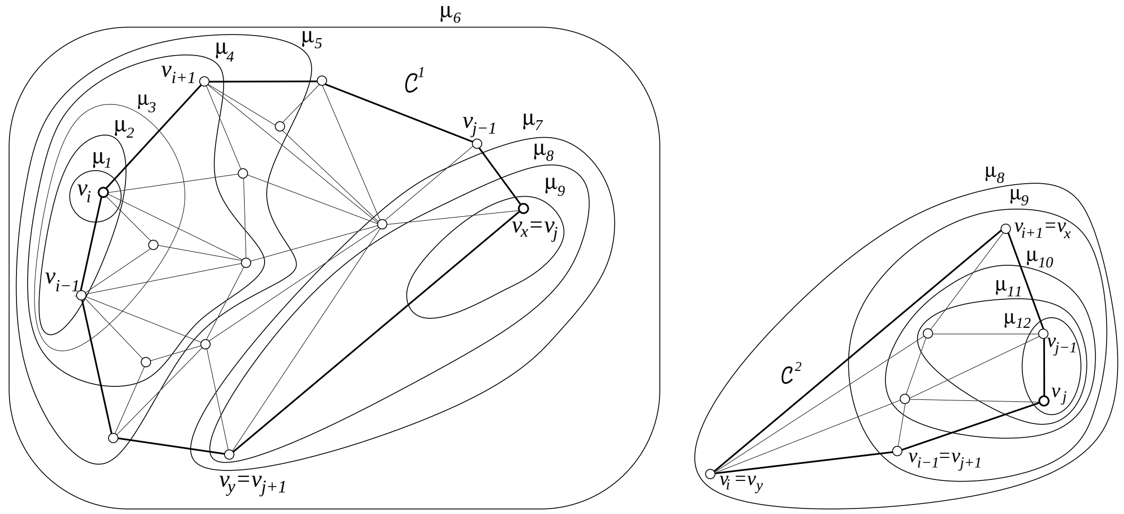
Let $C(G,T)$ be an outerclustered graph with outer face $f_o(G)$ delimited by a cycle $\mathcal{C} = (v_i, v_{i+1}, \dots, v_j, \dots, v_{i-1}, v_i)$. Suppose that C is linearly-ordered according to a sequence $\Sigma = \mu_1, \mu_2, \dots, \mu_k$ of clusters in T. Let (v_x, v_y) be a chord of $\mathcal{C}$. Consider the clustered graphs $C^1(G^1, T^1)$ and $C^2(G^2, T^2)$ such that G^1 (resp. G^2) is the subgraph of G induced by the vertices incident to and internal to cycle $\mathcal{C}^1 = (v_x, v_{x+1}, \dots, v_{y-1}, v_y, v_x)$ (resp. to cycle $\mathcal{C}^2 = (v_y, v_{y+1}, \dots, v_{x-1}, v_x, v_y)$), and such that T^1 (resp. T^2) is the subtree of T induced by the clusters containing vertices of G^1 (resp. of G^2).

Lemma 2 *$C^1(G^1, T^1)$ and $C^1(G^2, T^2)$ are linearly-ordered outerclustered graphs.*

Proof: We prove the statement for C^1, the proof for C^2 being analogous. Refer to Fig. 5. There exists no cluster μ in T^1 that contains vertices of C^1, that does not contain all the vertices of C^1, and that does not intersect $\mathcal{C}^1$ exactly twice. Namely, the border of every cluster μ is a simple closed curve, that hence intersects $\mathcal{C}^1$ an even number of times. Suppose that the border of μ does not intersect $\mathcal{C}^1$. Then, since μ contains vertices of G^1, it contains all of such vertices. Suppose that μ intersects $\mathcal{C}^1$ at least four times. At most two of such intersections can be on the edges of $\mathcal{C}^1 \setminus (v_x, v_y)$, since C is an outerclustered graph. Then, the border of μ intersects (v_x, v_y) at least twice, contradicting the c-planarity of C. By Lemma 1, it follows that C^1 is a biconnected internally-triangulated outerclustered graph.

Consider the subsequence Σ_1 of Σ induced by the clusters in T^1. Obtain a sequence Σ'_1 by removing from Σ_1 all the clusters that contain all the vertices of G^1 and that are different from $\sigma(v_x, v_{x+1}, \dots, v_y)$, if any such a cluster exists. We claim that C^1 is a linearly-ordered outerclustered graph according to Σ'_1.

We prove that C^1 satisfies Property LO1 of Definition 2. By definition, Σ'_1 contains all the clusters of T^1 that contain vertices of G^1, except for each cluster μ that contains all



(a) (b)

Figure 5: The linearly-ordered outerclustered graphs obtained by splitting the linearly-ordered outerclustered graph of Fig. 3 by chord (v_x, v_y). (a) $C^1(G^1, T^1)$. (b) $C^2(G^2, T^2)$.

the vertices of G^1 and that is different from $\sigma(v_x, v_{x+1}, \dots, v_y)$. However, for any vertex v of G^1, $\sigma(v) \neq \mu$, because $\sigma(v_x, v_{x+1}, \dots, v_y)$ is a descendant of μ and contains v.

We prove that C^1 satisfies Property LO2. Since C is linearly-ordered, $f_o(G)$ is delimited by two monotone paths $\mathcal{P}_1 = (v_i, v_{i+1}, \dots, v_j)$ and $\mathcal{P}_2 = (v_j, v_{j+1}, \dots, v_{i-1}, v_i)$. Suppose that (see Fig. 6.a) both v_x and v_y belong to $\mathcal{P}_1$ and v_x precedes v_y in $\mathcal{P}_1$ (the other cases in which v_x and v_y both belong to $\mathcal{P}_1$ or both belong to $\mathcal{P}_2$ being analogous); then, the subpath of $\mathcal{P}_1$ between v_x and v_y and edge (v_x, v_y) are monotone paths delimiting $f_o(G^1)$; further, $\mathcal{P}_2$ and the path obtained from $\mathcal{P}_1$ by replacing the subpath between v_x and v_y with edge (v_x, v_y) are monotone paths delimiting $f_o(G^2)$. Suppose that (see Fig. 6.b) v_x belongs to $\mathcal{P}_1$ and v_y belongs to $\mathcal{P}_2$; if $\sigma(v_x)$ precedes $\sigma(v_y)$ in Σ, then the subpath of $\mathcal{P}_1$ between v_x and v_j, and the subpath of $\mathcal{P}_2$ between v_j and v_y, augmented by edge (v_y, v_x), are monotone paths delimiting $f_o(G^1)$; if $\sigma(v_y)$ precedes $\sigma(v_x)$ in Σ, then the subpath of $\mathcal{P}_1$ between v_x and v_j, augmented by edge (v_y, v_x), and the subpath of $\mathcal{P}_2$ between v_j and v_y are monotone paths delimiting $f_o(G^1)$.

We prove that C^1 satisfies Property LO3. Since C is linearly-ordered according to Σ and the clusters in Σ'_1 also belong to Σ, so that if $\mu_x, \mu_y \in \Sigma'_1$ and $x < y$ then μ_x precedes μ_y in Σ, it suffices to show that any two consecutive clusters μ_x and μ_y of Σ'_1 are adjacent in Σ. Suppose, for a contradiction, that μ_y and μ_x are not adjacent in Σ. If μ_y is an ancestor of μ_x and is not its parent (the case in which μ_x is an ancestor of μ_y and is not its parent being analogous), consider the parent μ_{x+1} of μ_x. Such a cluster contains all the vertices contained in μ_x, hence either μ_{x+1} belongs to Σ'_1, contradicting the fact that μ_x and μ_y are consecutive in Σ'_1, or μ_{x+1} contains all the vertices of G^1 and is different from $\sigma(v_x, v_{x+1}, \dots, v_y)$. However, this would imply that also μ_y contains all the vertices of G^1 and is different from $\sigma(v_x, v_{x+1}, \dots, v_y)$, a contradiction to the fact that μ_y is in Σ'_1. Now suppose that μ_x and μ_y are incomparable, that is, μ_x is neither an ancestor nor a descendant of μ_y. Again, consider the parent μ_{x+1} of μ_x. Since μ_{x+1} is not in Σ'_1, then it contains all the vertices of G^1 and is different from $\sigma(v_x, v_{x+1}, \dots, v_y)$; this implies that $\mu_x = \sigma(v_x, v_{x+1}, \dots, v_y)$, hence μ_x is an ancestor of μ_y, a contradiction. □

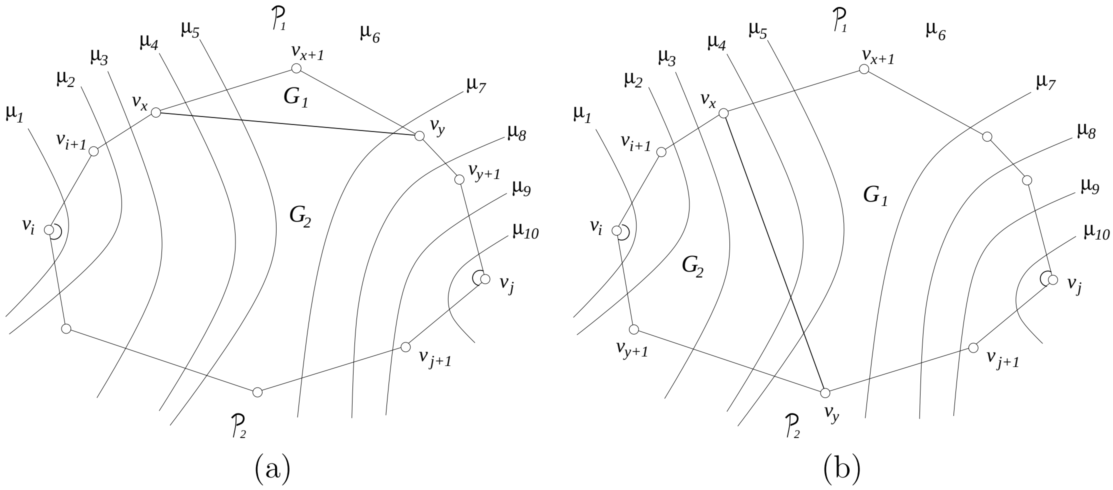


Figure 6: C^1 and C^2 satisfy LO2. (a) v_x and v_y belong to $\mathcal{P}_1$, and v_x precedes v_y in $\mathcal{P}_1$. (b) v_x belongs to $\mathcal{P}_1$, v_y belongs to $\mathcal{P}_2$, and $\sigma(v_x)$ precedes $\sigma(v_y)$ in Σ.

Let Γ be any convex-separated drawing of the outer face C_{f_o} of C. Suppose that v_x and v_y are not collinear with any vertex of $f_o(G)$. Let Γ_1 and Γ_2 be the drawings of $C^1_{f_o}$ and $C^2_{f_o}$ obtained by drawing (v_x, v_y) in Γ as a straight-line segment.

Lemma 3 *Γ_1 and Γ_2 are convex-separated drawings.*

Proof: We prove the statement for Γ_1, the proof for Γ_2 being analogous. Refer to Fig. 7. The drawing is straight-line and rectangular by construction. Since Γ is a convex-separated drawing, by Property CS1 of Definition 3, the polygon P_G representing $f_o(G)$ in Γ is convex. Further, by hypothesis, u_x and u_y are not collinear with any vertex of G. Hence, the polygon P_{G^1} representing $f_o(G^1)$ in Γ_1 is convex, thus satisfying Property CS1.

Drawing Γ_1 has no region-region crossings, since each cluster is represented in Γ_1 by the same rectangle as in Γ. We prove that Γ_1 has no edge-region crossings. Suppose that an edge-region crossing exists between an edge e and a cluster ν. Refer to Fig. 8. Then, $e = (v_x, v_y)$, otherwise Γ would not be c-planar. Cluster ν does not contain both of v_x and v_y otherwise, by the convexity of ν, e would be internal to ν; further, ν does not contain exactly one of v_x and v_y otherwise, by the convexity of ν, e would cross ν exactly once. It follows that ν does not contain neither v_x nor v_y. Consider the parent μ of ν in T_1. Such a parent exists otherwise ν would be the root of T, contradicting the fact that ν does not contain neither v_x nor v_y. Since Γ satisfies Property CS3, there exists a convex region $R(\mu, \nu)$ with the properties described in Definition 3; such a region separates ν from the rest of the drawing, thus avoiding an edge-region crossing between e and ν. More precisely, since Γ satisfies Property O2 of Definition 1, ν has exactly two incident edges $e_1(\nu)$ and $e_2(\nu)$ belonging to $f_o(G)$. Denote by $u(e_1(\nu))$ and $u(e_2(\nu))$ the endvertices of $e_1(\nu)$ and $e_2(\nu)$ belonging to ν. Denote by $p(l_1)$ and $p(l_2)$ the endpoints of $l_1(\mu, \nu)$ and $l_2(\mu, \nu)$ closer to $u(e_1(\nu))$ and $u(e_2(\nu))$, respectively. Then, by Property CS3, segment $\overline{p(l_1)p(l_2)}$ splits P_G into two convex polygons P_1 and P_2, where P_1 contains all and only the vertices in ν and P_2 contains all and only the vertices not in ν. By the convexity of P_1 and P_2, e is internal to P_2, while the part of ν inside P is internal to P_1. Hence, e does not cross ν.

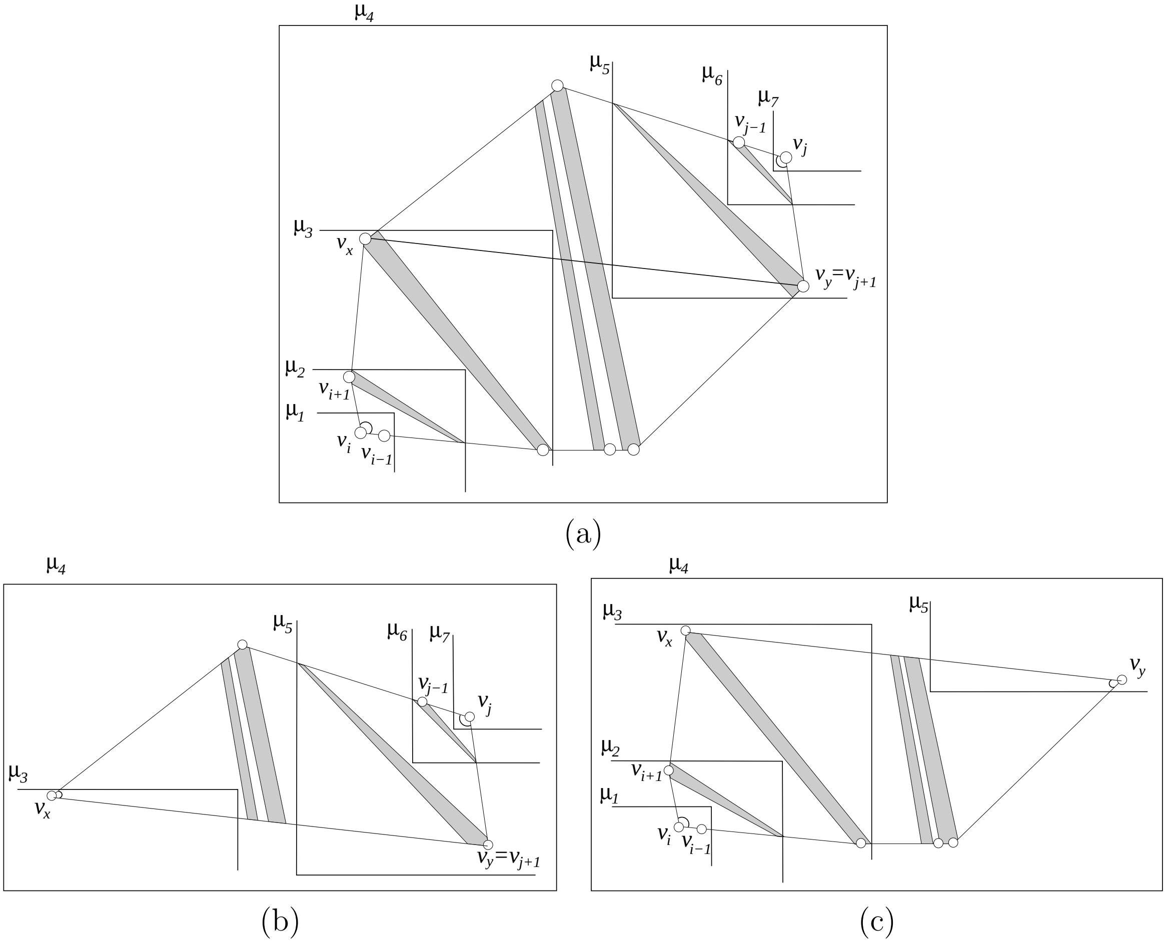


Figure 7: (a) A convex-separated drawing Γ of the outer face C_{f_o} of a linearly-ordered outerclustered graph C. (b) and (c) Convex-separated drawings Γ_1 and Γ_2 of the outer faces $C^1_{f_o}$ and $C^2_{f_o}$ of the linearly-ordered outerclustered graphs C^1 and C^2.

We prove that Γ_1 satisfies Property CS2. First, observe that the angles incident to v_i and v_j in P_{G^1} are strictly less than $180°$, since they are strictly less than $180°$ in Γ; further, the angles incident to v_x and v_y in P_{G^1} are strictly less than $180°$, since they are strictly less than the angles incident to v_x and v_y in P_G, that are at most $180°$, by the convexity of P_G. Suppose that $\sigma(v_x)$ precedes $\sigma(v_y)$ in Σ, the opposite case being analogous. Then, it suffices to observe that: (i) if C^1 contains both v_i and v_j, then $\Sigma'_1 = \Sigma$; hence, v_i and v_j are vertices satisfying the desired properties; (ii) if C^1 does not contain neither v_i nor v_j, then Σ'_1 is the subsequence of Σ that starts at $\sigma(v_x)$ and ends at $\sigma(v_y)$; hence, v_x and v_y are vertices satisfying the desired properties; (iii) if C^1 contains v_j and does not contain v_i, then Σ'_1 is the subsequence of Σ that starts at $\sigma(v_x)$ and ends at $\sigma(v_j) = \mu_k$; hence, v_x and v_j are vertices satisfying the desired properties.

We prove that Γ_1 satisfies Property CS3. The existence of regions $R(\mu, \nu)$ inside P_{G^1}, for every pair of clusters μ and ν in Σ'_1 such that μ is the parent of ν, is easily deduced from the existence of regions $R(\mu, \nu)$ inside P_G, which is guaranteed by Property CS3 of Definition 3. Namely, either $R(\mu, \nu)$ is not intersected by e, thus implying that a region $R(\mu, \nu)$ inside P_{G^1} can be constructed coincident with the same region inside P_G, or $R(\mu, \nu)$ is cut by e, thus creating two regions, one inside P_{G^1} and the other one inside

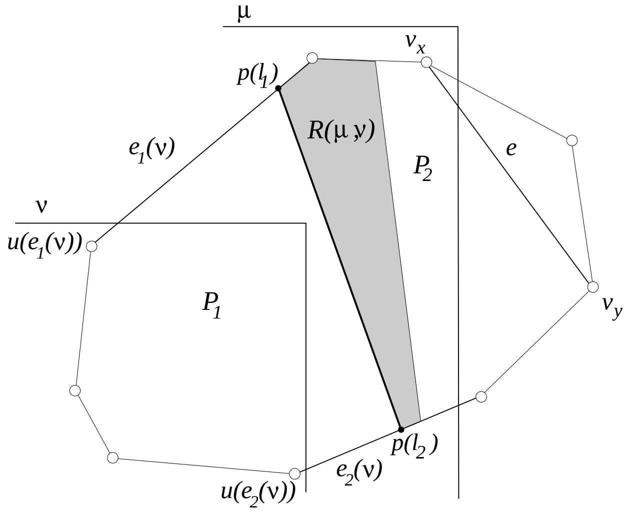


Figure 8: Drawing Γ_1 has no crossing between an edge e and a cluster containing none of the endvertices of e. The thick line represents segment $\overline{p(l_1)p(l_2)}$.

P_{G^2}. The properties that have to be satisfied by $R(\mu, \nu)$ inside P_{G^1} easily descend from the analogous properties satisfied by $R(\mu, \nu)$ inside P_G. □

When dealing with outerclustered graphs and general clustered graphs, it will be sufficient to consider clustered graphs whose underlying graphs have triangular outer faces. This leads us to the following refined definition:

Definition 4 *Let $C(G, T)$ be a clustered graph such that G is a 3-cycle (u, v, z). Denote by $e_1(\mu)$ and $e_2(\mu)$ the edges of G that are incident to any cluster μ of T not containing all the vertices of G. A straight-line rectangular drawing $\Gamma(C)$ of C is a triangular-convex-separated drawing if:*

- *TCS1: for every pair of clusters μ and ν such that μ is the parent of ν in T and such that μ is not an ancestor of $\sigma(u, v, z)$, there exists a convex region $R(\mu, \nu)$ such that: (i) $R(\mu, \nu)$ is entirely contained inside $\mu \cap (P \cup int(P))$, where P is the triangle representing G in $\Gamma(C)$; (ii) for any cluster $\mu' \neq \mu$ and any child ν' of μ', $R(\mu, \nu)$ does not intersect neither $R(\mu', \nu')$ nor the border of μ'; (iii) $R(\mu, \nu) \cap P$ consists of two continuous lines $l_1(\mu, \nu)$ and $l_2(\mu, \nu)$ such that at least one endpoint of $l_1(\mu, \nu)$ (resp. of $l_2(\mu, \nu)$) belongs to $e_1(\nu)$ (resp. to $e_2(\nu)$).*

Fig. 9 shows a triangular-convex-separated drawing of a clustered graph whose underlying graph is a 3-cycle.

The relationship between a triangular-convex-separated drawing of a clustered graph whose underlying graph is a 3-cycle and a convex-separated drawing of a linearly-ordered maximal outerclustered graph is clarified in the following lemma.

Lemma 4 *Let $C(G, T)$ be a linearly-ordered maximal outerclustered graph. Then, a triangular-convex-separated drawing of C_{f_o} is a convex-separated drawing of C_{f_o}.*

Proof: Consider any triangular-convex-separated drawing $\Gamma(C_{f_o})$ of C_{f_o}. We prove that $\Gamma(C_{f_o})$ is a convex-separated drawing. Properties CS1 and CS2 of Definition 3 easily

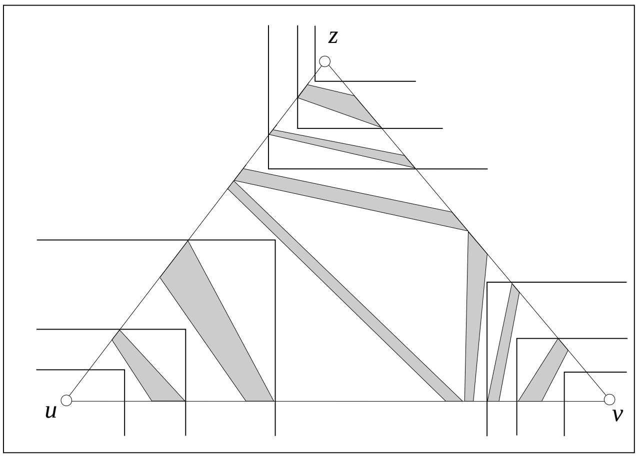


Figure 9: A triangular-convex-separated drawing of a clustered graph whose underlying graph is a 3-cycle.

descend from the fact that, since $f_o(G)$ is a 3-cycle, it is represented by a convex polygon P whose internal angles are smaller than 180°. Property CS3 coincides with Property TCS1. □

Finally, we extend the previous definition to general clustered graphs.

Definition 5 *Let $C(G,T)$ be an internally-triangulated clustered graph. A drawing $\Gamma(C)$ of C is an internally-convex-separated drawing if, for every internal face f of G, the part $\Gamma(C_f)$ of $\Gamma(C)$ representing C_f is a triangular-convex-separated drawing.*

3 Drawing Linearly-Ordered Outerclustered Graphs

In this section we show how to construct an internally-convex-separated drawing of any linearly-ordered outerclustered graph C for an arbitrary triangular-convex-separated drawing of the outer face C_{f_o} of C. This is done by means of an inductive algorithm that uses the following lemma as the main tool:

Lemma 5 *Let $C(G,T)$ be an internally-triangulated triconnected outerclustered graph. Suppose that C is linearly-ordered according to a sequence $\mu_1, \mu_2, \dots,$ μ_k of clusters of T. Let v_i and v_j be any two vertices such that $\sigma(v_i) = \mu_1$ and $\sigma(v_j) = \mu_k$. Let V_1 (resp. V_2) be the set of vertices between v_i and v_j (resp. between v_j and v_i) in the clockwise order of the vertices around $f_o(G)$. Then, if $V_1 \neq \emptyset$, there exists a path $\mathcal{P}_u = (u_1, u_2, \dots, u_r)$ such that (see Fig. 10):*

- *P1: u_1 and u_r belong to $V_2 \cup \{v_i, v_j\}$;*
- *P2: u_i is an internal vertex of G, for each $2 \leq i \leq r-1$;*
- *P3: if $\sigma(u_i) = \mu_{j_1}$ and $\sigma(u_{i+1}) = \mu_{j_2}$, then $j_1 < j_2$, for each $1 \leq i \leq r-1$;*
- *P4: there exists exactly one vertex u_x, where $2 \leq x \leq r-1$, that is adjacent to at least one vertex v_x in V_1;*

- *P5: there exist no chords among the vertices of path* $(u_1, u_2, \ldots, u_x)$ *and no chords among the vertices of path* $(u_x, u_{x+1}, \ldots, u_r)$.

Proof: We derive path $\mathcal{P}_u$ in several steps. At step $s+1$ a path $\mathcal{P}^{s+1}$ is found by modifying a path $\mathcal{P}^s$ obtained at the previous step. At the last step m of the algorithm, a path $\mathcal{P}_u$ can be obtained as a subpath of $\mathcal{P}^m$ satisfying the properties required by the lemma. A path satisfying properties P1–P5 is shown in Fig. 10.

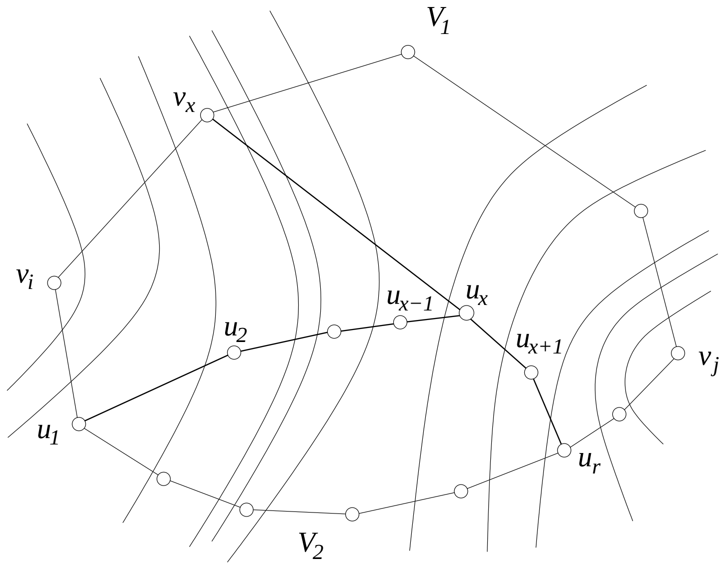


Figure 10: A path satisfying properties P1–P5.

More in detail, at each step $s < m$, a path $\mathcal{P}^s = (u^s_1, u^s_2, \ldots, u^s_{r(s)})$ is found satisfying the properties described below (see Fig. 11). Denote by $\mathcal{P}_1 = (v_i, v_{i+1}, v_{i+2}, \ldots, v_{j-1}, v_j)$ and $\mathcal{P}_2 = (v_i, v_{i-1}, v_{i-2}, \ldots, v_{j+1}, v_j)$ the monotone paths on the vertices of $V_1 \cup \{v_i, v_j\}$ and on the vertices of $V_2 \cup \{v_i, v_j\}$ delimiting $f_o(G)$, respectively.

- PP1: $u^s_1 = v_i$ and $u^s_{r(s)} = v_j$;
- PP2: each vertex u^s_t is either an internal vertex of G or a vertex of $\mathcal{P}_2$, for each $2 \leq t \leq r(s) - 1$;
- PP3: if $\sigma(u_t) = \mu_{j_1}$ and $\sigma(u_{t+1}) = \mu_{j_2}$, then $j_1 < j_2$, for each $1 \leq t \leq r(s) - 1$; and
- PP4: there exists no chord inside cycle $\mathcal{P}^s \cup \mathcal{P}_1$.

At the first step, set $\mathcal{P}^1 = \mathcal{P}_2$. Path $\mathcal{P}^1$ satisfies Property PP1 and PP2 by definition, Property PP3 since C satisfies Property LO2 of Definition 2, and Property PP4 because G is triconnected.

Suppose that a path $\mathcal{P}^s$ has been found at step s of the algorithm. We show how to determine $\mathcal{P}^{s+1}$ at step $s+1$ of the algorithm.

Consider any edge (u^s_t, u^s_{t+1}) of $\mathcal{P}^s$. Such an edge is incident to a face internal to cycle $\mathcal{P}^s \cup \mathcal{P}_1$. Let z^*_1 be the third vertex of such a face. By Property PP4, z^*_1 is neither a vertex of $\mathcal{P}_1$ nor a vertex of $\mathcal{P}^s$. Hence, z^*_1 is internal to $\mathcal{P}^s \cup \mathcal{P}_1$.

Denote by u^s_a and by u^s_b the first and the last vertex of $\mathcal{P}^s$ adjacent to z^*_1. We distinguish two cases.

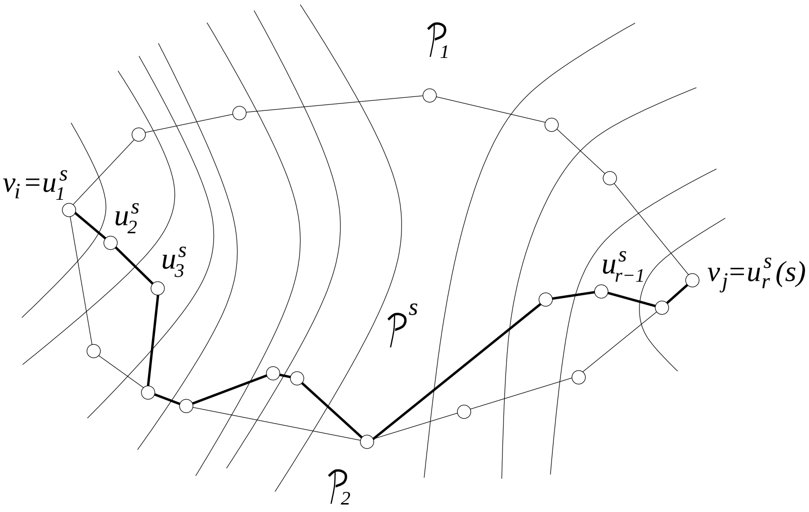


Figure 11: A path $\mathcal{P}^s$ satisfying properties PP1–PP4.

In the first case, $\sigma(u_a^s)$, $\sigma(z_1^*)$, and $\sigma(u_b^s)$ appear in this order in Σ. Then, path $\mathcal{P}^{s+1}$ is obtained by replacing the subpath of $\mathcal{P}^s$ between u_a^s and u_b^s with path (u_a^s, z_1^*, u_b^s). An example of this case is shown in Fig. 12.

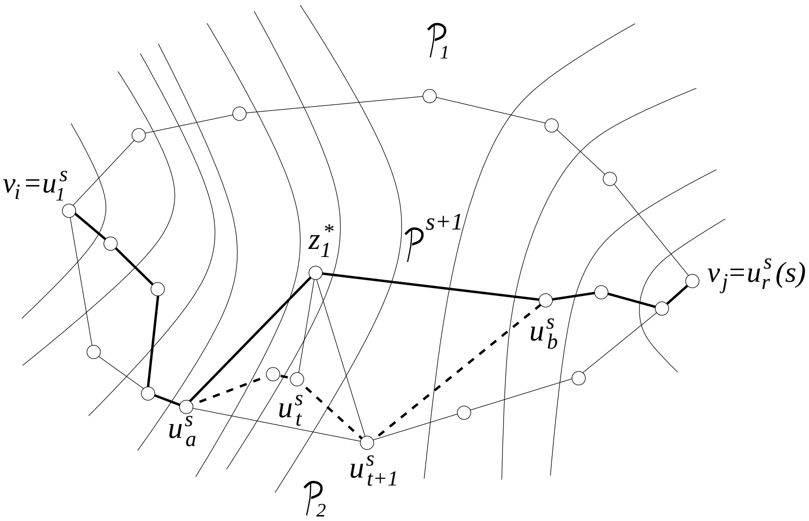


Figure 12: Obtaining $\mathcal{P}^{s+1}$ from $\mathcal{P}^s$ if $\sigma(u_a^s)$, $\sigma(z_1^*)$, and $\sigma(u_b^s)$ appear in this order in Σ. The dotted segments show the subpath of $\mathcal{P}^s$ that does not belong to $\mathcal{P}^{s+1}$.

Suppose that z_1^* is adjacent to at least one vertex in V_1. Since the first and the last vertex of $\mathcal{P}^{s+1}$ (that are v_i and v_j, respectively) belong to $\mathcal{P}_2$, the vertices shared by $\mathcal{P}^{s+1}$ and $\mathcal{P}_2$ partition $\mathcal{P}^{s+1}$ into subpaths. Let $\mathcal{P}_u$ be the one of such subpaths containing z_1^*. Path $\mathcal{P}_u = (u_1, u_2, \dots, u_x, u_{x+1}, \dots, u_r)$ defined as above is easily shown to satisfy properties P1–P4 (where $u_x = z_1^*$). Further, if there exists a chord (u_a, u_b) among the vertices of path $(u_1, u_2, \dots, u_x)$ (resp. among the vertices of path $(u_x, u_{x+1}, \dots, u_r)$), then replace the subpath of $(u_1, u_2, \dots, u_x)$ (resp. of $(u_x, u_{x+1}, \dots, u_r)$) between u_a and u_b with such a chord. The repetition of such an argument eventually leads to a path $\mathcal{P}_u$ also satisfying Property P5. Hence, the algorithm stops because a path with the properties required by the lemma has been found.

Suppose that z_1^* is not adjacent to any vertex in V_1. Then, $\mathcal{P}^{s+1}$ is easily shown to satisfy properties PP1–PP4. Hence, the algorithm continues with step $s+2$.

In the second case, either $\sigma(z_1^*)$, $\sigma(u_a^s)$, and $\sigma(u_b^s)$ appear in this order in Σ, or in the order $\sigma(u_a^s)$, $\sigma(u_b^s)$, $\sigma(z_1^*)$. Suppose that $\sigma(z_1^*)$, $\sigma(u_a^s)$, and $\sigma(u_b^s)$ appear in this order in Σ, the other case being analogous.

Consider edge (u_{a-1}^s, u_a^s) of $\mathcal{P}^s$. Such an edge is incident to a face internal to cycle $\mathcal{P}^s \cup \mathcal{P}_1$. Let z_2^* be the third vertex of such a face. By Property PP4, z_2^* is neither a vertex of $\mathcal{P}_1$ nor a vertex of $\mathcal{P}^s$. Hence, z_2^* is internal to $\mathcal{P}^s \cup \mathcal{P}_1$. Further, $\sigma(z_2^*)$ is not the same cluster of $\sigma(u_a^s)$ (since C satisfies Property O3 of Definition 1), and $\sigma(z_2^*)$ does not follow $\sigma(u_a^s)$ in Σ, otherwise edge (u_a^s, z_2^*) would cross twice the border of $\sigma(u_a^s)$ or the border of the cluster coming before $\sigma(u_a^s)$ in Σ (depending on whether $\sigma(u_a^s)$ is a child or is the parent of the cluster coming before $\sigma(u_a^s)$ in Σ). Hence, $\sigma(z_2^*)$ precedes $\sigma(u_a^s)$ in Σ.

Then, the whole argument can be repeated, namely denote by u_c^s and by u_d^s the first and the last vertex adjacent to z_2^* in $\mathcal{P}^s$. Since C satisfies Property O3, then $\sigma(u_c^s) \neq \sigma(z_2^*)$; since $\sigma(z_2^*)$ precedes $\sigma(u_a^s)$ in Σ and since $\sigma(u_a^s)$ does not follow $\sigma(u_d^s)$ in Σ, then $\sigma(z_2^*)$ precedes $\sigma(u_d^s)$ in Σ. Then, either $\sigma(u_c^s)$ precedes $\sigma(z_2^*)$ in Σ, or $\sigma(z_2^*)$ precedes $\sigma(u_c^s)$ in Σ.

If $\sigma(u_c^s)$ precedes $\sigma(z_2^*)$ in Σ, a path $\mathcal{P}^{s+1}$ is found by replacing the subpath of $\mathcal{P}^s$ between u_c^s and u_d^s with path (u_c^s, z_2^*, u_d^s); either $\mathcal{P}^{s+1}$ contains a path $\mathcal{P}_u$ satisfying properties P1–P5 or $\mathcal{P}^{s+1}$ satisfies properties PP1–PP4, depending on whether z_2^* is adjacent to at least one vertex in V_1 or not.

If $\sigma(z_2^*)$ precedes $\sigma(u_c^s)$ in Σ, edge (u_{c-1}^s, u_c^s) and the vertex z_3^* incident to the face $(u_{c-1}^s, u_c^s, z_3^*)$ internal to cycle $\mathcal{P}^s \cup \mathcal{P}_1$ are considered and the argument is repeated again. An example is depicted in Fig. 13. The repetition of such an argument eventually leads to find a vertex z_f^* that is incident to a face $(u_{y-1}^s, u_y^s, z_f^*)$ and such that, denoting by u_p^s and u_q^s the first and the last neighbor of z_f^* in $\mathcal{P}^s$, $\sigma(u_p^s)$, $\sigma(z_f^*)$, and $\sigma(u_q^s)$ appear in this order in Σ. Namely, at every repetition of such an argument, the considered edge (that is equal to (u_t^s, u_{t+1}^s) at the first repetition, to (u_{a-1}^s, u_a^s) at the second repetition, to (u_{c-1}^s, u_c^s) at the third repetition, and to (u_{y-1}^s, u_y^s) at the last repetition) gets closer to v_i in $\mathcal{P}^s$; then, after a certain number of repetitions of the algorithm, vertex u_{y-1}^s eventually belongs to cluster μ_1 and no cluster preceding μ_1 exists in Σ.

It remains to observe that, after a certain number m of steps of the algorithm, path $\mathcal{P}^m$ contains a path $\mathcal{P}_u$ satisfying properties P1–P5 as a subpath. Namely, the number of vertices internal to cycle $\mathcal{P}^s \cup \mathcal{P}_1$ decreases at every step of the algorithm, hence a vertex z^* adjacent to a vertex in V_1 is eventually added to a path $\mathcal{P}^{m-1}$ to form a path $\mathcal{P}^m$, that thus contains a path $\mathcal{P}_u$ satisfying properties P1–P5 as a subpath. Notice that z^* is the only vertex of $\mathcal{P}^m$ adjacent to a vertex in V_1, since no vertex of $\mathcal{P}^{m-1}$ is adjacent to a vertex in V_1, by Property PP4. □

A lemma similar to Lemma 5 is presented in the following. The proof of such a lemma can be obtained analogous to the one of Lemma 5, where V_1 replaces V_2, and vice versa.

Lemma 6 *Let $C(G,T)$ be an internally-triangulated triconnected outerclustered graph. Suppose that C is linearly-ordered according to a sequence $\mu_1, \mu_2, \dots, \mu_k$ of clusters of T. Let v_i and v_j be any two vertices such that $\sigma(v_i) = \mu_1$ and $\sigma(v_j) = \mu_k$. Let V_1 (resp. V_2) be the set of vertices between v_i and v_j (resp. between v_j and v_i) in the clockwise order of the vertices around $f_o(G)$. Then, if $V_1 \neq \emptyset$, there exists a path $\mathcal{P}_u = (u_1, u_2, \dots, u_r)$ such that:*

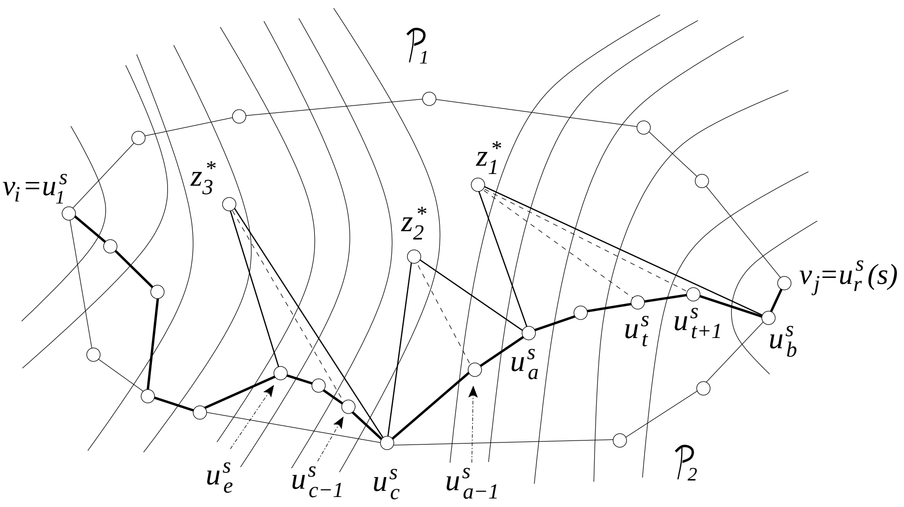


Figure 13: Three repetitions of the argument for the case in which $\sigma(z_1^*)$ (resp. $\sigma(z_2^*)$ and $\sigma(z_3^*)$) precedes the smallest clusters containing the first and last neighbor of z_1^* (resp. of z_2^* or of z_3^*) in $\mathcal{P}^s$. Thick segments represent $\mathcal{P}^s$.

- *P1: u_1 and u_r belong to $V_2 \cup \{v_i, v_j\}$;*
- *P2: u_i is an internal vertex of G, for each $2 \leq i \leq r-1$;*
- *P3: if $\sigma(u_i) = \mu_{j_1}$ and $\sigma(u_{i+1}) = \mu_{j_2}$, then $j_1 < j_2$, for each $1 \leq i \leq r-1$;*
- *P4: there exists exactly one vertex u_x, where $2 \leq x \leq r-1$, that is adjacent to at least one vertex v_x in V_1;*
- *P5: there exist no chords among the vertices of path $(u_1, u_2, \ldots, u_x)$ and no chords among the vertices of path $(u_x, u_{x+1}, \ldots, u_r)$.*

We now present the main theorem of this section.

Theorem 1 *Let $C(G,T)$ be a linearly-ordered internally-triangulated triconnected outer-clustered graph. Then, for every convex-separated drawing $\Gamma(C_{f_o})$ of C_{f_o}, there exists an internally-convex-separated drawing $\Gamma(C)$ of C completing $\Gamma(C_{f_o})$.*

Proof: We prove the statement by induction on the number of internal vertices of G. First observe that, since G is triconnected, $f_o(G)$ has no chords. Hence, if G has no internal vertices, then C_{f_o} and C are the same graph and the statement trivially follows. Otherwise, G has internal vertices. Denote by $\mathcal{C}$ the cycle delimiting $f_o(G)$. We show how to split C into smaller linearly-ordered internally-triangulated triconnected outerclustered graphs.

Let $\Gamma(C_{f_o})$ be an arbitrary convex-separated drawing of C_{f_o}. Denote by P the polygon representing $f_o(G)$ in $\Gamma(C_{f_o})$. Since $\Gamma(C_{f_o})$ satisfies Property CS1 of Definition 3, P is convex. Suppose that C is linearly-ordered according to a sequence $\Sigma = \mu_1, \mu_2, \ldots, \mu_h, \ldots, \mu_k$ of clusters.

Let v_i and v_j be two vertices of $\mathcal{C}$ such that $\sigma(v_i) = \mu_1$ and $\sigma(v_j) = \mu_k$, and such that the angle of P incident to v_i and the angle of P incident to v_j are strictly less than 180°. Such vertices exist since $\Gamma(C_{f_o})$ satisfies Property CS2 of Definition 3.

We distinguish two cases:

Case 1 applies when the vertices of $V_1 \cup \{v_i, v_j\}$ are not all collinear. In such a case, $V_1 \neq \emptyset$, otherwise the only two vertices of $V_1 \cup \{v_i, v_j\}$ would be collinear. Hence, a path $(u_1, u_2, \ldots, u_x, \ldots, u_r)$ can be found as in Lemma 5. Let v_x be any vertex of V_1 adjacent to u_x.

Lemma 7 *Vertex v_x does not lie on the line $l(u_1, u_r)$ through u_1 and u_r.*

Proof: Denote by $\mathcal{P}(u_1, v_x, u_r)$ the subpath of $\mathcal{C}$ between vertices u_1 and u_r containing v_x. If there exists a vertex of $\mathcal{P}(u_1, v_x, u_r)$ lying on the segment $\overline{u_1 u_r}$, then all the vertices in $\mathcal{P}(u_1, v_x, u_r)$ lie on $\overline{u_1 u_r}$, otherwise polygon P would not be convex (see Fig. 14.a). However, $\mathcal{P}(u_1, v_x, u_r)$ contains all the vertices of $V_1 \cup \{v_i, v_j\}$, that are not all collinear, by hypothesis. Now suppose that v_x lies on the half-line that lies on $l(u_1, u_r)$, that starts at u_1, and that does not contain u_r (the case in which v_x lies on the half-line that lies on $l(u_1, u_r)$, that starts at u_r, and that does not contain u_1 being analogous). By the convexity of P, all the vertices in the path between v_x and u_r containing u_1 lie on $l(u_1, u_r)$ (see Fig. 14.b). However, since $v_x \in V_1$, since $u_1, u_r \in V_2 \cup v_i, v_j$, and since v_i and v_j are the endvertices of $\mathcal{P}_2$, it follows that v_i is one of the internal vertices of the path between v_x and u_r containing u_1, thus its incident angle in P is exactly 180°, contradicting the hypothesis. □

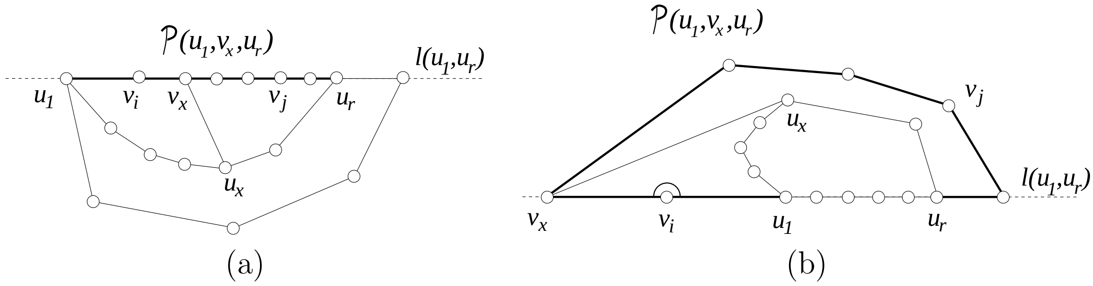


Figure 14: Illustration for the proof of Lemma 7. (a) Vertex v_x lies on $\overline{u_1 u_r}$. (b) Vertex v_x lies on the half-line that lies on $l(u_1, u_r)$, that starts at u_1, and that does not contain u_r.

Consider the triangle $T(v_x, u_1, u_r)$ with vertices v_x, u_1, and u_r. By Lemma 7, such a triangle is non-degenerate and, by the convexity of P, it is entirely contained inside P (parts of the borders of $T(v_x, u_1, u_r)$ and P could coincide). Denote by $\sigma'(u_i)$ any cluster in Σ child of $\sigma(u_i)$, for each $2 \leq i \leq r-1$.

Lemma 8 *There exists a small disk D contained inside $int(T(v_x, u_1, u_r)) \cap R(\sigma(u_x), \sigma'(u_x))$.*

Proof: Let μ_{j_1}, μ_{j_2}, μ_{j_3}, and μ_{j_4} be clusters $\sigma(u_1)$, $\sigma(u_x)$, $\sigma(u_r)$, and $\sigma(v_x)$ in Σ. Denote by μ'_j any cluster in Σ child of μ_j. Since $2 \leq x \leq r-1$ and since $\sigma(u_i) = \mu_{l_1}$ and $\sigma(u_{i+1}) = \mu_{l_2}$ implies $l_1 < l_2$, we have $j_1 < j_2 < j_3$. Since C satisfies Property O3 of Definition 1, $\sigma(u_x) \neq \sigma(v_x)$ and hence $j_2 \neq j_4$. Suppose that $j_2 < j_4$, the case $j_2 > j_4$ being analogous.

We claim that $s_1 = R(\mu_{j_2}, \mu'_{j_2}) \cap \overline{u_1 u_r}$ and $s_2 = R(\mu_{j_2}, \mu'_{j_2}) \cap \overline{u_1 v_x}$ are straight-line segments. The claim implies the lemma, namely if s_1 and s_2 are straight-line segments, then the interior of the quadrilateral having s_1 and s_2 as opposite sides entirely belongs to

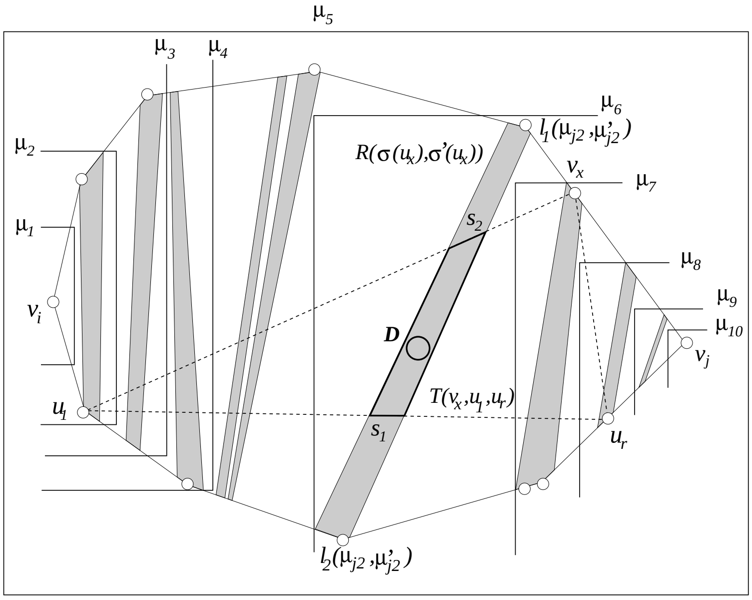


Figure 15: Illustration for the proof of Lemma 8.

$int(T(v_x, u_1, u_r)) \cap R(\mu_{j_2}, \mu'_{j_2})$ and any disk entirely contained inside such a quadrilateral satisfies the requirements of the lemma. See Fig. 15.

By definition of convex-separated drawing, $R(\mu_{j_2}, \mu'_{j_2}) \cap P$ consists of two line segments $l_1(\mu_{j_2}, \mu'_{j_2})$ and $l_2(\mu_{j_2}, \mu'_{j_2})$. Since $j_2 \neq j_1, j_3, j_4$, then none of vertices u_1, u_r, and v_x belongs to $R(\mu_{j_2}, \mu'_{j_2})$. Consider the two line segments l_a and l_b obtained by removing $l_1(\mu_{j_2}, \mu'_{j_2})$ and $l_2(\mu_{j_2}, \mu'_{j_2})$ from P. Since P is convex, it suffices to show that u_1 is in l_a and u_r is in l_b, or vice versa, in order to prove that s_1 is a straight-line segment, and that u_1 is in l_a and v_x is in l_b, or vice versa, in order to prove that s_2 is a straight-line segment. However, this directly follows from the fact that one of the sides of the border of region $R(\mu_{j_2}, \mu'_{j_2})$ separates the vertices whose smallest containing cluster is μ_j, with $j < j_2$, from the vertices whose smallest containing cluster is μ_j, with $j > j_2$. □

Place u_x at any point inside D. Draw edge (u_x, v_x) as a straight-line segment. Consider the straight-line segment $\overline{u_1 u_x}$ and consider any cluster μ_j such that $j_1 < j < j_2$. Then, $R(\mu_j, \mu'_j) \cap \overline{u_1 u_x}$ is a straight-line segment, namely one of the sides of the border of region $R(\mu_j, \mu'_j)$ separates the vertices whose smallest containing cluster is $\mu_{j'}$, with $j' < j$, from the vertices whose smallest containing cluster is $\mu_{j'}$, with $j' > j$. Analogously, consider the straight-line segment $\overline{u_x u_r}$ and consider any cluster μ_j such that $j_2 < j < j_3$. Then, $R(\mu_j, \mu'_j) \cap \overline{u_x u_r}$ is a straight-line segment, namely one of the sides of the border of region $R(\mu_j, \mu'_j)$ separates the vertices whose smallest containing cluster is $\mu_{j'}$, with $j' < j$, from the vertices whose smallest containing cluster is $\mu_{j'}$, with $j' > j$. Draw each vertex u_i, with $2 \leq i \leq x - 1$, at any internal point of $\overline{u_1 u_x} \cap R(\sigma(u_i), \sigma'(u_i))$. Analogously, draw each vertex u_i, with $x + 1 \leq i \leq r - 1$, at any point of $\overline{u_x u_r} \cap R(\sigma(u_i), \sigma'(u_i))$. Denote by Γ the constructed drawing. The construction of Γ is depicted in Fig. 16.

Let $\mathcal{C}_1$ be the cycle composed of path $(u_1, u_2, \ldots, u_x)$, of edge (u_x, v_x), and of the path between u_1 and v_x in $\mathcal{C}$ not containing u_r. Let $\mathcal{C}_2$ be the cycle composed of path $(u_x, u_{x+1}, \ldots, u_r)$, of edge (u_x, v_x), and of the path between v_x and u_r in $\mathcal{C}$ not containing u_1. Let $\mathcal{C}_3$ be the cycle composed of path $(u_1, u_2, \ldots, u_x, \ldots, u_r)$ and of the path between u_1 and u_r in $\mathcal{C}$ not containing v_x. Denote by G_1, G_2, and G_3 the subgraphs of G induced

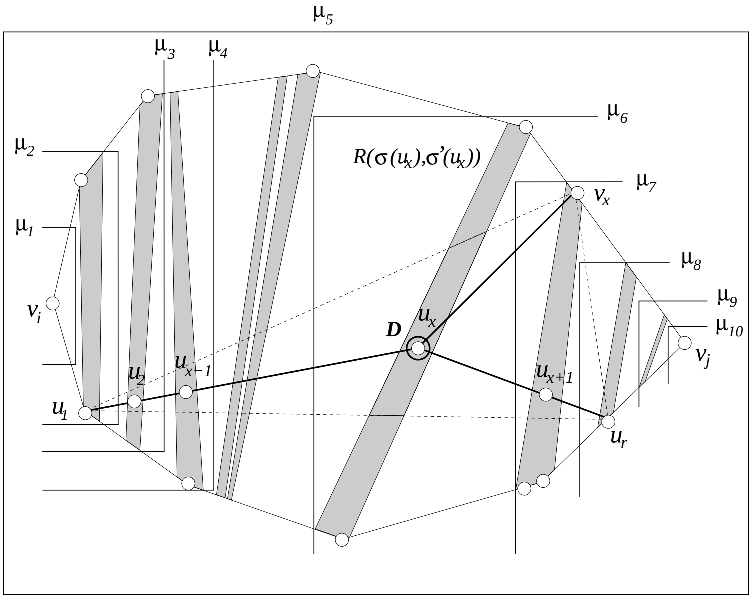


Figure 16: Drawing path $(u_1, u_2, \ldots, u_x, \ldots, u_r)$ and edge (u_x, v_x).

by the vertices incident to or internal to $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$, respectively. Finally, let C_1, C_2, and C_3 be the clustered graphs whose underlying graphs are G_1, G_2, and G_3, respectively, and whose inclusion trees T_1, T_2, and T_3 are the subtrees of T induced by the clusters containing vertices of G_1, G_2, and G_3, respectively.

Lemma 9 *C_1, C_2, and C_3 are linearly-ordered outerclustered graphs.*

Proof: Denote by $\mathcal{C}_{1,2}$ the cycle composed of path $(u_1, u_2, \ldots, u_x, \ldots, u_r)$ and of the path between u_1 and u_r in $\mathcal{C}$ containing v_x. Let $C_{1,2}$ be the clustered graph whose underlying graph $G_{1,2}$ is the subgraph of G whose vertices are incident to or internal to $\mathcal{C}_{1,2}$, and whose inclusion tree $T_{1,2}$ is the subtree of T induced by the clusters containing vertices of $G_{1,2}$. It suffices to prove that $C_{1,2}$ and C_3 are linearly-ordered outerclustered graphs. Namely, by Lemma 2, if $C_{1,2}$ is a linearly-ordered outerclustered graph, then C_1 and C_2 are linearly-ordered outerclustered graphs, as well.

Since C is linearly-ordered, $f_o(G)$ is delimited by two monotone paths $\mathcal{P}_1$ and $\mathcal{P}_2$.

Consider sequence $\Sigma = \mu_1, \mu_2, \ldots, \mu_k$, that is, the sequence according to which C is linearly-ordered. Denote by Σ_3 the subsequence of Σ that starts at $\sigma(u_1)$ and ends at $\sigma(u_r)$. We prove that $C_{1,2}$ is an outerclustered graph linearly-ordered according to Σ and that C_3 is an outerclustered graph linearly-ordered according to Σ_3.

Each cluster μ_i that is not crossed by path $(u_1, u_2, \ldots, u_x, \ldots, u_r)$ has two intersections with $\mathcal{C}_{1,2}$ and none with $\mathcal{C}_3$. Consider any cluster μ_i that is crossed by path $(u_1, u_2, \ldots, u_x, \ldots, u_r)$. Since C satisfies O2, the border $B(\mu_i)$ of each cluster μ_i intersects $\mathcal{C}$ exactly twice. One of such intersections belongs to $\mathcal{C}_{1,2}$, the other one to $\mathcal{C}_3$. Hence, each of $\mathcal{C}_{1,2}$ and $\mathcal{C}_3$ crosses $B(\mu_i)$ exactly twice. Then, Lemma 1 ensures that $C_{1,2}$ and C_3 are biconnected internally-triangulated outerclustered graphs.

We prove that $C_{1,2}$ and C_3 satisfy Property LO2 of Definition 2. Path $\mathcal{P}_1$ and the path obtained by replacing the subpath of $\mathcal{P}_2$ between u_1 and u_r with path $(u_1, u_2, \ldots, u_x, \ldots, u_r)$ are monotone paths delimiting $f_o(G_{1,2})$. Path $(u_1, u_2, \ldots, u_x, \ldots, u_r)$ and the subpath of $\mathcal{P}_2$ between u_1 and u_r are monotone paths delimiting $f_o(G_3)$.

We prove that $C_{1,2}$ and C_3 satisfy Property LO1. Since C is linearly-ordered, the smallest cluster containing each vertex in G (and each vertex in $G_{1,2}$) is a cluster μ_i, that by hypothesis belongs to Σ. Concerning C_3, observe that $f_o(G_3)$ is delimited by two monotone paths. Hence, if the smallest cluster containing a vertex in G_3 does not belong to Σ_3, then such a vertex is outside the cycle $f_o(G_3)$, a contradiction.

We prove that $C_{1,2}$ and C_3 satisfy Property LO3. Sequence Σ for $C_{1,2}$ coincides with sequence Σ for C, hence the property follows. Concerning C_3, observe that the existence of an index h such that μ_{i+1} is the parent of μ_i, for each $i < h$ such that μ_i belongs to Σ_3, and such that μ_{i+1} is a child of μ_i, for each $h \leq i$ such that μ_{i+1} belongs to Σ_3, easily descends from the existence of such an index for the sequence Σ of C and from the fact that Σ_3 is a subsequence of Σ. $\square$

Now we turn our attention to the geometry supporting the above topological results.

Lemma 10 *The constructed drawings of* $(\mathcal{C}_1, T_1)$, $(\mathcal{C}_2, T_2)$, *and* $(\mathcal{C}_3, T_3)$ *are convex-separated drawings.*

Proof: Drawing Γ is straight-line and rectangular by construction. By construction, u_x lies in $int(T(v_x, u_1, u_r))$ that in turn is entirely contained in $int(P)$. By the convexity of P, straight-line segments can be drawn from u_x to any vertex of P (and hence to u_1, v_x, and u_r) not causing crossings; hence, the drawings of $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ have no edge crossings. Since $\Gamma(C_{f_o})$ has no region-region crossings, Γ has no region-region crossings (namely, each cluster has the same drawing in Γ and in $\Gamma(C_{f_o})$).

We prove that Γ has no edge-region crossings. Suppose that there is an edge-region crossing between an edge e and a cluster ν. Then, e is either an edge of path $(u_1, u_2, \dots, u_x)$, or an edge of path $(u_x, u_{x+1}, \dots, u_r)$, or edge (u_x, v_x). Namely, all other edge-cluster pairs have the same drawings in Γ and in $\Gamma(C_{f_o})$, hence they do not cross more than once by hypothesis. Suppose that $e = (u_i, u_{i+1})$ is an edge of $(u_1, u_2, \dots, u_x)$, the other cases being analogous. If both u_i and u_{i+1} belong to ν then, by the convexity of ν, e is internal to ν. If exactly one of u_i and u_{i+1} belongs to ν then, by the convexity of ν, e crosses ν exactly once. It follows that ν does not contain neither u_i nor u_{i+1}. Consider the parent μ of ν in T. Such a parent exists otherwise ν would be the root of T, contradicting the fact that ν does not contain neither u_i nor u_{i+1}. By definition of convex-separated drawing, there exists a convex region $R(\mu, \nu)$ with the properties described in Definition 3; such a region separates ν from the rest of the drawing, thus avoiding an edge-region crossing between e and ν. More precisely, by definition of outerclustered graph, ν has exactly two incident edges $e_1(\nu)$ and $e_2(\nu)$ belonging to $f_o(G)$. Denote by $u(e_1(\nu))$ and $u(e_2(\nu))$ the endvertices of $e_1(\nu)$ and $e_2(\nu)$ belonging to ν. Denote by $p(l_1)$ and $p(l_2)$ the endpoints of $l_1(\mu, \nu)$ and $l_2(\mu, \nu)$ closer to $u(e_1(\nu))$ and $u(e_2(\nu))$, respectively. Then, segment $\overline{p(l_1)p(l_2)}$ splits P in two convex polygons P_a and P_b, where P_a contains all and only the vertices of $f_o(G)$ and of path $(u_1, u_2, \dots, u_x)$ in ν and P_b contains all and only the vertices of $f_o(G)$ and of path $(u_1, u_2, \dots, u_x)$ not belonging to ν. By convexity, e is internal to P_b, while the part of ν inside P is internal to P_a. Hence, e does not cross ν.

We prove that Γ satisfies Property CS1 of Definition 3. Denote by P_i the polygon representing $\mathcal{C}_i$ in Γ, for each $i \in \{1, 2, 3\}$. Every angle that is incident to a vertex in $\mathcal{C}$ different from u_1, u_r, and v_x in P_1, P_2, or P_3 is no more than 180°, since it is the same angle as in P. Every angle that is incident to a vertex in the path $(u_1, u_2, \dots, u_r)$ different from u_1, u_x, and u_r in P_1, P_2, or P_3 is exactly 180°, by construction. Every angle incident to u_1, v_x, and u_r in P_1, P_2, or P_3 is strictly less than 180°, since it is strictly less than an angle that is at most 180° (namely the angle incident to the same vertex in P); finally, the three angles $\widehat{u_1 u_x v_x}$, $\widehat{u_1 u_x u_r}$, and $\widehat{u_r u_x v_x}$ incident to u_x are all strictly less than 180°, since they are angles in triangles $T(u_1 u_x v_x)$, $T(u_1 u_x u_r)$, and $T(u_r u_x v_x)$, respectively.

We prove that Γ satisfies Property CS2. Concerning the drawing of $\mathcal{C}_1$ (the arguments for the drawing of $\mathcal{C}_2$ being analogous), observe that C_1 is linearly-ordered according to the subsequence Σ_1 of Σ that starts at $\sigma(v_i) = \mu_1$ and ends at the one of $\sigma(u_x)$ and $\sigma(v_x)$ that comes after in Σ. The angle incident to v_i in P_1 is strictly less than 180°, by hypothesis; further, as proved above, the angles incident to u_x and v_x in P_1 are strictly less than 180°. Concerning the drawing of $\mathcal{C}_3$, observe that $\sigma(u_1)$ and $\sigma(u_r)$ are the first and the last cluster in Σ_3. Further, the angles incident to u_1 and to u_r in P_3 are strictly smaller than 180°, as proved above.

We prove that Γ satisfies Property CS3. Let μ_j be any non-minimal cluster belonging to the sequence with respect to which C_i is linearly-ordered and let μ'_j be any child of μ_j. For each $i = 1, 2, 3$, the existence of a region $R(\mu_j, \mu'_j)$ inside P_i is easily deduced from the existence of region $R(\mu_j, \mu'_j)$ inside P. Namely, either $R(\mu_j, \mu'_j)$ inside P is not intersected by path $(u_1, u_2, \dots, u_r)$ nor by edge (u_x, v_x), implying that a region $R(\mu_j, \mu'_j)$ inside P_i can be constructed coincident with the same region inside P, or $R(\mu_j, \mu'_j)$ inside P is intersected by one or both of path $(u_1, u_2, \dots, u_r)$ and edge (u_x, v_x), that thus split $R(\mu_j, \mu'_j)$ in two or three regions inside the polygons P_i. The properties that have to be satisfied by $R(\mu_j, \mu'_j)$ inside P_i easily descend from the analogous properties satisfied by $R(\mu_j, \mu'_j)$ inside P. □

Graphs C_1, C_2, and C_3 are, in general, not triconnected; namely, there could exist chords in C_1 and in C_2 between u_x and any vertex in the path of $\mathcal{C}$ connecting u_1 and u_r and containing v_x; there could exist chords in C_3 between any vertex in $(u_1, u_2, \dots, u_{x-1})$ and any vertex in $(u_{x+1}, u_{x+2}, \dots, u_r)$, and between any vertex in $(u_1, u_2, \dots, u_r)$ and any vertex in the path of $\mathcal{C}$ connecting u_1 and u_r and not containing v_x. By Lemma 2, each of these chords splits a linearly-ordered outerclustered graph into two smaller linearly-ordered outerclustered graphs. Further, by construction the endvertices of each of such chords are not collinear with any other vertex of the cycle. Hence, by Lemma 3, inserting the chords as straight-line segments into the drawings of C_1, C_2, and C_3, that are convex-separated by Lemma 10, splits them into convex-separated drawings. When all chords have been added, the underlying graphs of the resulting linearly-ordered outerclustered graphs are all triconnected and internally-triangulated. Hence, the induction applies and an internally-convex-separated drawing of each of such linearly-ordered clustered graphs can be constructed inside the corresponding outer face, thus obtaining an internally-convex-separated drawing of C.

Case 2 applies when the vertices of $V_2 \cup \{v_i, v_j\}$ are not all collinear. In such a case, $V_2 \neq \emptyset$ and a path $(u_1, u_2, \dots, u_x, \dots, u_r)$ can be found as in Lemma 5bis. Analogously to Case 1, C is decomposed into smaller triconnected internally-triangulated linearly-ordered outerclustered graphs; path $(u_1, u_2, \dots, u_x, \dots, u_r)$ and all the edges connecting vertices of such a path with vertices of $\mathcal{C}$ can be drawn so that the outer faces of the corresponding outerclustered graphs are represented by convex-separated drawings.

It remains to prove that one out of Case 1 and Case 2 can always be applied. If Case 1 does not apply, then the vertices of $V_1 \cup \{v_i, v_j\}$ are collinear, and if Case 2 does not apply, then the vertices of $V_2 \cup \{v_i, v_j\}$ are collinear. However, this would imply that all the vertices of $\mathcal{C}$ are collinear, contradicting the fact that $\Gamma(C_{f_o})$ is a convex-separated drawing. This concludes the proof of Theorem 1. □

4 Drawing Outerclustered Graphs

In this section we generalize from linearly-ordered outerclustered graphs to general outerclustered graphs. In order to show that any maximal outerclustered graph has an internally-convex-separated drawing completing an arbitrary triangular-convex-separated drawing of its outer face, we show how to reduce the problem of drawing an outerclustered graph to the one of drawing some linearly-ordered outerclustered graphs.

First, we need the following preliminary results.

Lemma 11 *Let $C(G,T)$ be a maximal outerclustered graph and let u, v, and z be the vertices incident to $f_o(G)$. Let $u_i \neq u$ be any internal vertex of G such that $\sigma(u_i)$ contains u and does not contain neither v nor z. Then, there exists an edge (u_i, u_{i+1}) in G such that $\sigma(u_{i+1})$ is a descendant of $\sigma(u_i)$.*

Proof: Refer to a c-planar embedding of C and to Fig. 17. Let μ_{i+1} be the only cluster child of $\sigma(u_i)$ and containing u. Notice that such a child exists. Namely, if such a child does not exist, then $\sigma(u_i) = \sigma(u)$; this would imply that $u_i = u$, since $\sigma(u)$ induces a connected graph, by the c-planarity and the maximality of C, and since an outerclustered graph does not contain two adjacent vertices having the same smallest containing cluster if one of them is internal, by Property O3 of Definition 1. Suppose, for a contradiction, that u_i is not adjacent to any vertex in μ_{i+1}. By Property O3, u_i is not adjacent to any vertex v_j such that $\sigma(v_j) = \sigma(u_i)$. It follows that each edge e_l incident to u_i intersects the border $B(\sigma(u_i))$ of $\sigma(u_i)$ in a point p_l belonging to the line segment $l(\sigma(u_i))$ that is the part of $B(\sigma(u_i))$ lying in the interior of $f_o(G)$. Order the points p_l as they are encountered walking on $l(\sigma(u_i))$ from one endpoint to the other one. Denote by $p_1, p_2, \ldots, p_m$ such points. By the maximality of G, $m \geq 3$ and there exists an internal face delimited by a 3-cycle containing edges e_1 and e_m. However, the third edge e^* of such a cycle crosses twice $l(\sigma(u_i))$ and hence $B(\sigma(u_i))$, contradicting the c-planarity of the considered embedding. □

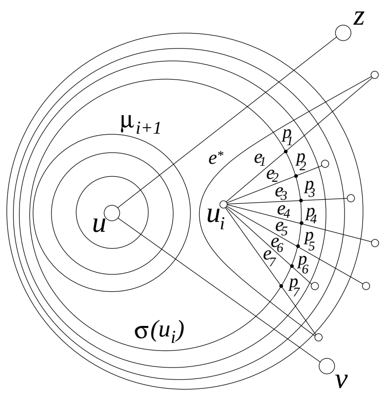


Figure 17: Illustration for the proof of Lemma 11.

Corollary 1 *Let $C(G, T)$ be any maximal outerclustered graph and let u, v, and z be the vertices incident to $f_o(G)$. Suppose that $\sigma(u) \neq \sigma(v), \sigma(z)$. Let u_1 be any internal vertex of G such that $\sigma(u_1)$ contains u and does not contain neither v nor z. Then, there exists a chordless path $(u_1, u_2, \ldots, u_k)$ in G such that $u_k = u$ and such that $\sigma(u_{i+1})$ is a descendant of $\sigma(u_i)$, for each $i = 1, 2, \ldots, k-1$.*

Proof: Suppose that a path $(u_1, u_2, \ldots, u_i)$ has already been determined such that u_{j+1} is a descendant of u_j, for each $j = 1, 2, \ldots, i-1$. By Lemma 11, there exists an edge (u_i, u_{i+1}) in G such that $\sigma(u_{i+1})$ is a descendant of $\sigma(u_i)$. Repeating such an argument eventually leads to choose a vertex u_k in $\sigma(u)$. Since $\sigma(u) \neq \sigma(v), \sigma(z)$, then u is the only vertex in $\sigma(u)$. It follows that $u_k = u$. The obtained path $\mathcal{P}_u$ may have chords. Suppose that the currently considered path $\mathcal{P}_u^l = (u_1^l, u_2^l, \ldots, u_k^l)$ has a chord (u_i^l, u_j^l), with $j > i+1$ and with $\mathcal{P}_u^1 = \mathcal{P}_u$. Obtain a new path $\mathcal{P}_u^{l+1}$ by replacing the subpath $(u_i^l, u_{i+1}^l, \ldots, u_j^l)$ of $\mathcal{P}_u^l$ with the chord (u_i^l, u_j^l). Clearly, $\sigma(u_j^l)$ is a descendant of $\sigma(u_i^l)$. Further, $\mathcal{P}_u^{l+1}$ has at least one chord less than $\mathcal{P}_u^l$ which implies that after a certain number of steps, the current path is chordless. □

Let $C(G, T)$ be a maximal outerclustered graph and let u, v, and z be the vertices incident to $f_o(G)$. Suppose that $\sigma(u) \neq \sigma(v)$, $\sigma(u) \neq \sigma(z)$, and $\sigma(v) \neq \sigma(z)$. Suppose also that, if there exists a cluster containing exactly two vertices incident to $f_o(G)$, then such vertices are u and v. Suppose also that, if the smallest containing cluster of one of u and v contains the other one, then $\sigma(v)$ contains u. Finally, suppose that G has internal vertices.

Lemma 12 *One of the following holds:*

1. *There exist three paths $\mathcal{P}_u = (u_1, u_2, \ldots, u_U)$, $\mathcal{P}_v = (v_1, v_2, \ldots, v_V)$, and $\mathcal{P}_z = (z_1, z_2, \ldots, z_Z)$ such that (see Fig. 18):*

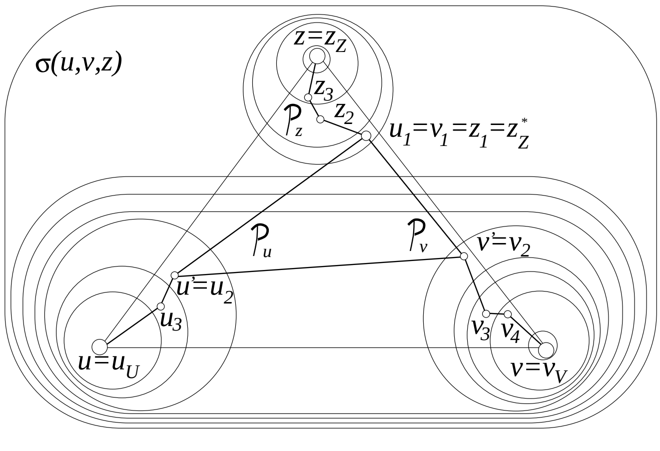


Figure 18: Three paths $\mathcal{P}_u$, $\mathcal{P}_v$, and $\mathcal{P}_z$ satisfying Condition 1 of Lemma 12.

   (a) $u_U = u$, $v_V = v$, $z_Z = z$, *and* $u_1 = v_1 = z_1$;

   (b) the vertices of $\mathcal{P}_u \setminus \{u_1\}$, $\mathcal{P}_v \setminus \{v_1\}$, *and* $\mathcal{P}_z \setminus \{z_1\}$ *are distinct;*

   (c) each of paths $\mathcal{P}_u \setminus \{u_1\}$, $\mathcal{P}_v \setminus \{v_1\}$, *and* $\mathcal{P}_z$ *has no chords;*

(d) $\sigma(u_i)$ *does not contain neither* v *nor* z*, for each* $2 \leq i \leq U$*;* $\sigma(v_i)$ *does not contain neither* u *nor* z*, for each* $2 \leq i \leq V$*;* $\sigma(z_i)$ *does not contain neither* u *nor* v*, for each* $Z^* \leq i \leq Z$*, where* Z^* *is an index such that* $1 \leq Z^* \leq Z$*;*

(e) $\sigma(u_{i+1})$ *is a descendant of* $\sigma(u_i)$*, for each* $2 \leq i \leq U-1$*;* $\sigma(v_{i+1})$ *is a descendant of* $\sigma(v_i)$*, for each* $2 \leq i \leq V-1$*;*

(f) $\sigma(z_1)$ *is either a cluster containing* z *and not containing* u *and* v *(then* $Z^* = 1$ *and* $\sigma(z_{i+1})$ *is a descendant of* $\sigma(z_i)$*, for each* $1 \leq i \leq Z-1$*), or is* $\sigma(u, v, z)$ *(then* $Z^* = 2$ *and* $\sigma(z_{i+1})$ *is a descendant of* $\sigma(z_i)$*, for each* $1 \leq i \leq Z-1$*), or a cluster not containing* z *and containing* u *and* v*. In the latter case* $Z^* \geq 2$*,* $\sigma(z_{i+1})$ *is an ancestor of* $\sigma(z_i)$*, for each* $1 \leq i \leq Z^*-2$*,* $\sigma(z_{i+1})$ *is a descendant of* $\sigma(z_i)$*, for each* $Z^* \leq i \leq Z-1$*, and either* $\sigma(z_{Z^*})$ *is a descendant of* $\sigma(z_{Z^*-1})$ *(if* $\sigma(z_{Z^*-1}) = \sigma(u, v, z)$*), or* $\sigma(z_{Z^*})$ *is not comparable with* $\sigma(z_{Z^*-1})$ *(if* $\sigma(z_{Z^*-1})$ *contains* u *and* v *and does not contain* z*).*

(g) G *contains an internal face having incident vertices* u_1*,* u_2*, and* v_2*.*

2. *There exist two paths* $\mathcal{P}_u = (u_1, u_2, \ldots, u_U)$ *and* $\mathcal{P}_v = (v_1, v_2, \ldots, v_V)$ *such that:*

(a) $u_U = u$*,* $v_V = v$*, and* $u_1 = v_1 = z$*;*

(b) *the vertices of* $\mathcal{P}_u \setminus \{u_1\}$ *and* $\mathcal{P}_v \setminus \{v_1\}$ *are distinct;*

(c) *each of paths* $\mathcal{P}_u \setminus \{u_1\}$ *and* $\mathcal{P}_v \setminus \{v_1\}$ *has no chords;*

(d) $\sigma(u_i)$ *does not contain neither* v *nor* z*, for each* $2 \leq i \leq U$*;* $\sigma(v_i)$ *does not contain neither* u *nor* z*, for each* $2 \leq i \leq V$*;*

(e) $\sigma(u_{i+1})$ *is a descendant of* $\sigma(u_i)$*, for each* $2 \leq i \leq U-1$*;* $\sigma(v_{i+1})$ *is a descendant of* $\sigma(v_i)$*, for each* $2 \leq i \leq V-1$*;*

(f) G *contains an internal face having incident vertices* u_2*,* v_2*, and* z*.*

Proof: Consider the biggest cluster μ_u containing u and not containing v. Notice that such a cluster exists. Namely, $\sigma(u)$ does not contain v by hypothesis. Consider the biggest cluster μ_v containing v and not containing u, if any such a cluster exists. If μ_v exists, let E' be the set of edges whose end-vertices are one in μ_u and one in μ_v. If μ_v does not exist, let E' be the set of edges incident to v whose other end-vertex is in μ_u. In both cases, there exists at least one of such edges, in fact (u, v). Order the edges in E' as they are encountered when walking on the part of the border $B(\mu_u)$ of μ_u that is internal to G, starting from the intersection of (u, v) with $B(\mu_u)$. See Fig. 19.

Consider the last edge (u', v') in E' and consider the internal face (u', v', z') such that (u', z') is the edge following (u', v') in the order in which the edges incident to μ_u are encountered when walking on the part of $B(\mu_u)$ that is internal to G, starting from the intersection of (u, v) with $B(\mu_u)$. Let $\mathcal{P}_u = (u_1, u_2, \ldots, u_U)$ be a path such that $u_1 = z'$, $u_2 = u'$, and $(u_2, u_3, \ldots, u_U)$ is a chordless path such that $u_U = u$ and $\sigma(u_{i+1})$ is a descendant of $\sigma(u_i)$, for $2 \leq i \leq U-1$. Such a path exists by Corollary 1 (notice that $\sigma(u_2)$ contains u and does not contain neither v nor z). Further, let $\mathcal{P}_v = (v_1, v_2, \ldots, v_V)$ be a path such that $v_1 = z'$, $v_2 = v'$, and $(v_2, v_3, \ldots, v_V)$ is a chordless path such that $v_V = v$ and $\sigma(v_{i+1})$ is a descendant of $\sigma(v_i)$, for $2 \leq i \leq V-1$. Notice that if μ_v exists, then such a path exists by Corollary 1 (notice that $\sigma(v_2)$ contains v and does not contain neither u nor z). Otherwise, $v_2 = v_V = v$.

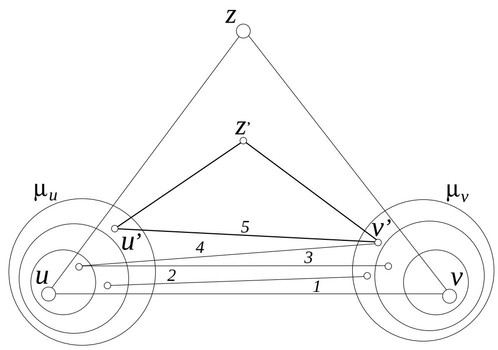


Figure 19: The edges whose endvertices are one in μ_u and one in μ_v. The numbers on such edges indicate the order in which such edges are encountered when walking on the part of $B(\mu_u)$ internal to G, starting from the intersection of (u, v) with $B(\mu_u)$.

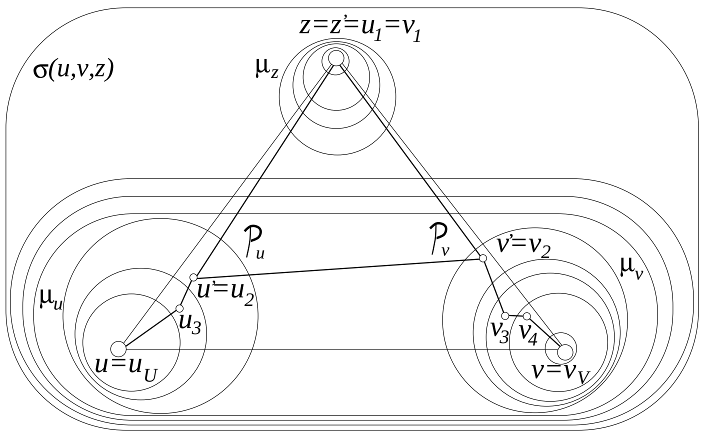


Figure 20: If $z' = z$, then paths $\mathcal{P}_u$ and $\mathcal{P}_v$ satisfy Condition 2 of Lemma 12.

Suppose that $z' = z$ (see Fig. 20). Then, G contains an internal face having incident vertices u_2, v_2, and z. By construction, paths $\mathcal{P}_u$ and $\mathcal{P}_v$ satisfy Condition 2 of the lemma.

Suppose that $z \neq z'$. Vertex z' does not belong neither to μ_u nor to μ_v, otherwise edge (z', v') or edge (z', u') would follow (u', v') in E', a contradiction. Hence, $\sigma(z')$ is either a cluster containing z and not containing u and v, or is $\sigma(u, v, z)$, or is a cluster containing u and v and not containing z.

- Suppose that $\sigma(z')$ contains z and does not contain u and v (see Fig. 21). Let $\mathcal{P}_z = (z_1, z_2, \ldots, z_Z)$ be a chordless path such that $z_1 = z'$, $z_Z = z$, and z_{i+1} is a descendant of z_i, for $2 \leq i \leq Z - 1$. Such a path exists by Corollary 1 (notice that $\sigma(z_1)$ contains z and does not contain neither u nor v). By construction, paths $\mathcal{P}_u$, $\mathcal{P}_v$, and $\mathcal{P}_z$ satisfy Condition 1 of the lemma.

- Suppose that $\sigma(z') = \sigma(u, v, z)$. Refer to a c-planar embedding of C. Let μ_z be the biggest cluster containing z and not containing neither u nor v. We claim that there exists an edge incident to z' and to a vertex belonging to μ_z. Suppose, for a contradiction, that no edge incident to z' is incident to a vertex belonging to μ_z. Vertex z' is not adjacent to any vertex v_j such that $\sigma(v_j) = \sigma(u, v, z)$, by Property O3 of Definition 1. Hence, for each neighbor v_j of z', either $\sigma(v_j)$ contains

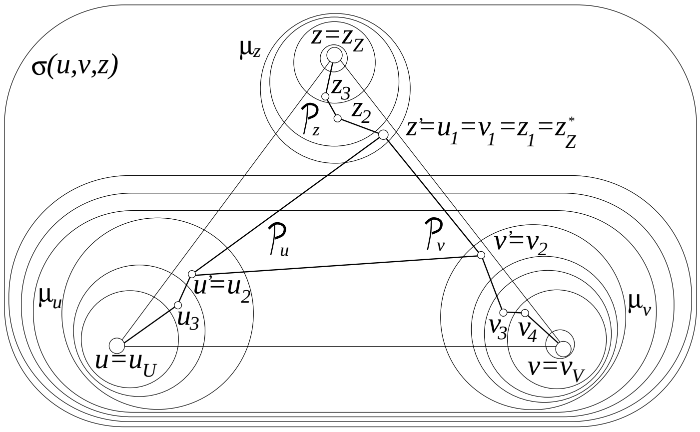


Figure 21: If $z' \neq z$ and $\sigma(z')$ contains z and does not contain u and v, then paths $\mathcal{P}_u$, $\mathcal{P}_v$, and $\mathcal{P}_z$ satisfy Condition 1 of Lemma 12.

u and does not contain v and z, or $\sigma(v_j)$ contains v and does not contain u and z, or $\sigma(v_j)$ contains u and v and does not contain z.

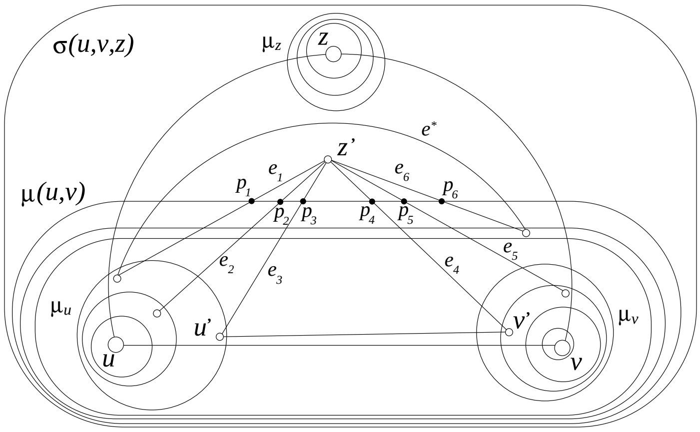


Figure 22: If $z' \neq z$, if $\sigma(z') = \sigma(u, v, z)$, and if a cluster containing both u and v and not containing z exists, then an edge incident to z' and to a vertex belonging to μ_z exists.

First, suppose that a cluster containing both u and v and not containing z exists (see Fig. 22). Let $\mu_{u,v}$ be the biggest cluster containing both u and v and not containing z. Then, each edge e_l incident to z' intersects the border $B(\mu_{u,v})$ of $\mu_{u,v}$ in a point p_l belonging to the line segment $l(\mu_{u,v})$ that is the part of $B(\mu_{u,v})$ lying in the interior of $f_o(G)$. Order points p_l as they are encountered walking on $l(\mu_{u,v})$ from one endpoint to the other one. Denote by $p_1, p_2, \dots, p_m$ such points. By the maximality of G, $m \geq 3$ and there exists a face delimited by a 3-cycle containing edges e_1 and e_m. However, the third edge e^* of such a cycle crosses twice $l(\mu_{u,v})$, and hence $B(\mu_{u,v})$, contradicting the c-planarity of the considered embedding.

Second, suppose that a cluster containing both u and v and not containing z does not exist (see Fig. 23). Then, each edge e_l incident to z' is also incident either to a vertex belonging to μ_u or to a vertex belonging to μ_v. Notice that there exists at least one neighbor of z' in μ_u, namely u', and at least one neighbor of z' in μ_v, namely v'. Hence, by the c-planarity of the embedding, the order of the edges

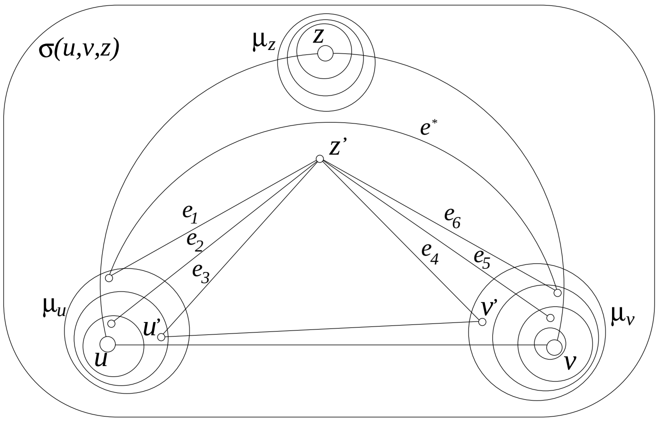


Figure 23: If $z' \neq z$, if $\sigma(z') = \sigma(u, v, z)$, and if a cluster containing both u and v and not containing z does not exist, then an edge incident to z' and to a vertex belonging to μ_z exists.

incident to z' is a sequence $(e_1, e_2, \ldots, e_i, e_{i+1}, \ldots, e_m)$ so that: (i) e_j is incident to a vertex in μ_u, for $j = 1, 2, \ldots, i$, (ii) $e_i = (z', u')$, (iii) $e_{i+1} = (z', v')$, and (iv) e_j is incident to a vertex in μ_v, for $j = i+1, i+2, \ldots, m$. By the maximality of G, $m \geq 3$ and there exists a face delimited by a 3-cycle containing edges e_1 and e_m. However, the third edge e^* of such a cycle is an edge between a vertex in μ_u and a vertex in μ_v, contradicting the fact that (u', v') is the last edge in E'.

Then, there exists an edge (z', z_2) incident to z' such that z_2 belongs to μ_z. Let $(z_2, z_3, \ldots, z_Z)$ be a chordless path such that $z_Z = z$ and z_{i+1} is a descendant of z_i, for $2 \leq i \leq Z - 1$. Such a path exists by Corollary 1 (notice that $\sigma(z_2)$ contains z and does not contain neither u nor v). Consider path $(z_1, z_2, z_3, \ldots, z_Z)$. For each chord (z_1, z_j), with $j > 2$, replace the subpath of the current path by edge (z_1, z_j). This results in a chordless path $\mathcal{P}_z$. By construction, paths $\mathcal{P}_u$, $\mathcal{P}_v$, and $\mathcal{P}_z$ satisfy Condition 1 of the lemma.

- Suppose that $\sigma(z')$ contains u and v and does not contain z. In the following we show how to find a path $\mathcal{P}_z^1 = (z_1, z_2, \ldots, z_{Z^*})$ such that $z_1 = z'$, $\sigma(z_i)$ contains u and v and does not contain z, for $1 \leq i \leq Z^* - 2$, $\sigma(z_{i+1})$ is an ancestor of $\sigma(z_i)$, for each $1 \leq i \leq Z^* - 2$, z_{Z^*} belongs to μ_z, and either $\sigma(z_{Z^*})$ is a descendant of $\sigma(z_{Z^*-1})$ (if $\sigma(z_{Z^*-1}) = \sigma(u, v, z)$), or $\sigma(z_{Z^*})$ is not comparable with $\sigma(z_{Z^*-1})$ (if $\sigma(z_{Z^*-1})$ contains u and v and does not contain z).

  Suppose that $\mathcal{P}_z^1$ has already been determined till a vertex z_i. We claim that there exists an edge incident to z_i and to a vertex z_{i+1} such that either $\sigma(z_{i+1})$ is an ancestor of $\sigma(z_i)$ containing u and v and not containing z, or $\sigma(z_{i+1}) = \sigma(u, v, z)$, or $\sigma(z_{i+1})$ contains z and does not contain u and v.

  Suppose, for a contradiction, that every edge incident to z_i is incident to a vertex belonging to μ_u, or to a vertex belonging to μ_v, or to a vertex belonging to a cluster that contains u and v, that does not contain z, and that is a descendant of $\sigma(z_i)$. Vertex z_i is not adjacent to any vertex v_j such that $\sigma(v_j) = \sigma(z_i)$, by Property O3 of Definition 1.

  Suppose that $\sigma(z_i) \neq \sigma(u, v)$, that is, there exists a child $\mu(u, v, \neg z_i)$ of $\sigma(z_i)$ in

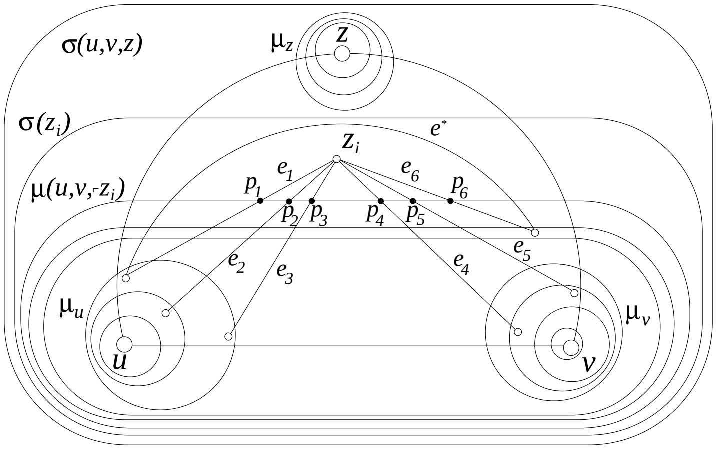


Figure 24: If $z' \neq z$, if $\sigma(z')$ contains u and v and does not contain z, and if $\mathcal{P}_z^1$ has already been determined till a vertex z_i, then an edge incident to z_i and to a vertex z_{i+1} exists such that either $\sigma(z_{i+1})$ is an ancestor of $\sigma(z_i)$ containing u and v and not containing z, or $\sigma(z_{i+1}) = \sigma(u, v, z)$, or $\sigma(z_{i+1})$ contains z and does not contain u and v.

T containing u and v and not containing z (see Fig. 24). Each edge incident to z_i is also incident to a vertex belonging to $\mu(u, v, \neg z_i)$. Analogously as above, a contradiction can be reached by proving that there exists an edge e^* that crosses twice the border $B(\mu(u, v, \neg z_i))$ of $\mu(u, v, \neg z_i)$.

Suppose that $\sigma(z_i) = \sigma(u, v)$. Then, a contradiction can be reached by proving that: (i) if all the edges incident to z_i are also incident to vertices belonging to μ_u (or if all the edges incident to z_i are also incident to vertices belonging to μ_v), then there exists an edge that crosses twice the border $B(\mu_u)$ of μ_u (resp. the border $B(\mu_v)$ of μ_v); (ii) if some edges incident to z_i are also incident to vertices in μ_u and some edges incident to z_i are also incident to vertices in μ_v, then (u', v') is not the last edge in E'.

This proves the claim, namely that there exists an edge incident to z_i and to a vertex z_{i+1} such that either $\sigma(z_{i+1})$ is an ancestor of $\sigma(z_i)$ containing u and v and not containing z, or $\sigma(z_{i+1}) = \sigma(u, v, z)$, or $\sigma(z_{i+1})$ contains z and does not contain u and v. The repetition of such an argument eventually leads to the choice of a vertex z_{Z^*} that belongs to μ_z or to the choice of a vertex z_{Z^*-1} such that $\sigma(z_{Z^*-1}) = \sigma(u, v, z)$. If z_{Z^*} belongs to μ_z, then $\mathcal{P}_z^1$ has already been determined satisfying the desired properties. If $\sigma(z_{Z^*-1}) = \sigma(u, v, z)$ then the same arguments as above show that there exists an edge (z_{Z^*-1}, z_{Z^*}) such that z_{Z^*} belongs to μ_z, thus obtaining a path $\mathcal{P}_z^1$ satisfying the desired properties.

Let $\mathcal{P}_z^2 = (z_{Z^*}, z_{Z^*+1}, \dots, z_Z)$ be a chordless path such that $z_Z = z$, and z_{i+1} is a descendant of z_i, for $Z^* \leq i \leq Z - 1$. Such a path exists by Corollary 1 (notice that $\sigma(z_{Z^*})$ contains z and does not contain neither u nor v).

Finally, consider the path obtained by concatenating $\mathcal{P}_z^1$ and $\mathcal{P}_z^2$. Such a path may eventually have chords. For each chord (z_i, z_j), with $j > i + 1$, replace the subpath of the current path by edge (z_i, z_j). This results in a chordless path $\mathcal{P}_z$. By construction, paths $\mathcal{P}_u$, $\mathcal{P}_v$, and $\mathcal{P}_z$ satisfy Condition 1 of the lemma.

□

Suppose that Condition 1 of Lemma 12 holds. Denote by $C_{u,v}$, by $C_{u,z}$, and by $C_{v,z}$ the clustered graphs whose underlying graphs $G_{u,v}$, $G_{u,z}$, and $G_{v,z}$ are the subgraphs of G induced by the vertices incident to and internal to cycles $\mathcal{C}_{u,v} \equiv (u,v) \cup (\mathcal{P}_u \setminus \{u_1\}) \cup (u_2,v_2) \cup (\mathcal{P}_v \setminus \{v_1\})$, $\mathcal{C}_{u,z} \equiv (u,z) \cup \mathcal{P}_u \cup \mathcal{P}_z$, and $\mathcal{C}_{v,z} \equiv (v,z) \cup \mathcal{P}_v \cup \mathcal{P}_z$, and whose inclusion trees $T_{u,v}$, $T_{u,z}$, and $T_{v,z}$ are the subtrees of T induced by the clusters containing vertices of $G_{u,v}$, $G_{u,z}$, and $G_{v,z}$, respectively. Fig. 25 shows graphs $C_{u,v}$, $C_{u,z}$, and $C_{v,z}$.

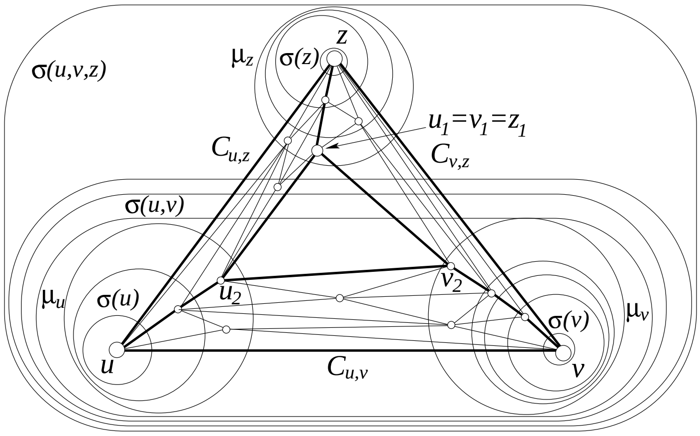


Figure 25: Graphs $C_{u,v}$, $C_{u,z}$, and $C_{v,z}$ when Condition 1 of Lemma 12 holds. Thick edges show paths $\mathcal{P}_u$, $\mathcal{P}_v$, $\mathcal{P}_z$, and edges (u,v), (u,z), (v,z), and (u_2,v_2).

Lemma 13 *$C_{u,v}$, $C_{u,z}$, and $C_{v,z}$, are linearly-ordered outerclustered graphs.*

Proof: We prove the statement for $C_{u,v}$, the proofs for $C_{u,z}$ and $C_{v,z}$ being analogous. Refer to a c-planar embedding of $C_{u,v}$ and to Fig. 26. The outer face of $G_{u,v}$ is delimited by a simple cycle $\mathcal{C}_{u,v}$. We prove that the border of each cluster μ containing vertices of $\mathcal{C}_{u,v}$ and not containing all the vertices of $\mathcal{C}_{u,v}$ intersects $\mathcal{C}_{u,v}$ exactly twice. By construction, each cluster containing both u and v contains all the vertices of $G_{u,v}$ and each cluster not containing neither u nor v does not contain any vertex of $G_{u,v}$. Finally, each cluster μ containing u and not containing v (the arguments for each cluster μ containing v and not containing u being analogous) intersects edge (u,v) exactly once, intersects path $(\mathcal{P}_u \setminus \{u_1\}) \cup (u_2,v_2)$ exactly once, and does not intersect path $\mathcal{P}_v \setminus \{v_1\}$. It follows that μ intersects $\mathcal{C}_{u,v}$ twice. By Lemma 1, $C_{u,v}$ is a biconnected internally-triangulated outerclustered graph.

Consider the following sequence of clusters $\Sigma = \mu_1, \mu_2, \dots, \mu_k$, where $\mu_1 = \sigma(u)$, μ_{i+1} is the parent of μ_i in $T_{u,v}$, for $i = 1, 2, \dots, h-2$, where μ_{h-1} is the biggest cluster containing u and not containing v, μ_h is $\sigma(u,v)$, μ_{h+1} is the biggest cluster containing v and not containing u, μ_{i+1} is the only child of μ_i in $T_{u,v}$ containing v, for $i = h+1, h+2, \dots, k-1$, and $\mu_k = \sigma(v)$. In the following we prove that $C_{u,v}$ is linearly-ordered according to Σ.

We prove that $C_{u,v}$ satisfies Property LO1 of Definition 2. Each vertex x in $G_{u,v}$ either belongs to μ_{h-1}, or to μ_{h+1}, or is such that $\sigma(x) = \sigma(u,v)$. Namely, all the vertices incident to $f_o(G_{u,v})$ either belong to μ_{h-1} or to μ_{h+1}, by construction; then, all the vertices of $G_{u,v}$ belong to the smallest cluster containing both μ_{h-1} and μ_{h+1}, that is $\sigma(u,v)$. Further, Σ includes all and only the clusters containing at least one out of u and v, with the exception of the clusters containing both u and v and ancestors of $\sigma(u,v)$; however, each cluster containing both u and v, and ancestor of $\sigma(u,v)$ is different from $\sigma(x)$, for any vertex x

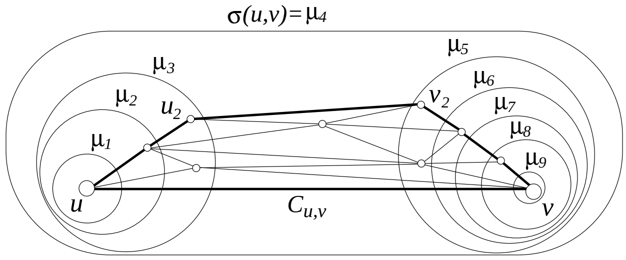


Figure 26: A closer look to $C_{u,v}$. Cluster sequence Σ is shown.

of $G_{u,v}$, since x is also contained in $\sigma(u,v)$. Since, for each internal vertex x of G, $\sigma(x)$ is a cluster containing at least one out of u and v, then $\sigma(x) = \mu_i$, for some $1 \leq i \leq k$.

We prove that $C_{u,v}$ satisfies Property LO2. Edge (u,v) and the path obtained by concatenating $\mathcal{P}_u \setminus \{u_1\}$, (u_2, v_2), and $\mathcal{P}_v \setminus \{v_1\}$ are monotone paths delimiting $f_o(G_{u,v})$.

We prove that $C_{u,v}$ satisfies Property LO3. By construction, μ_{i+1} is the parent of μ_i, for $i = 1, 2, \ldots, h-2$, and μ_{i+1} is a child of μ_i, for $i = h+1, h+2, \ldots, k-1$. Hence, in order to prove that $C_{u,v}$ satisfies Property LO3, it suffices to show that μ_h is the parent of μ_{h-1} and that μ_h is the parent of μ_{h+1}. By construction, μ_{h-1} (resp. μ_{h+1}) is the biggest cluster containing u and not containing v (resp. containing v and not containing u). Hence, the parent of μ_{h-1} (resp. of μ_{h+1}) is $\sigma(u,v)$, that by definition is μ_h. □

Suppose that Condition 2 of Lemma 12 holds. Denote by $C_{u,v}$, by $C_{u,z}$, and by $C_{v,z}$ the clustered graphs whose underlying graphs $G_{u,v}$, $G_{u,z}$, and $G_{v,z}$ are the subgraphs of G induced by the vertices incident to and internal to cycles $\mathcal{C}_{u,v} \equiv (u,v) \cup (\mathcal{P}_u \setminus \{u_1\}) \cup (u_2, v_2) \cup (\mathcal{P}_v \setminus \{v_1\})$, $\mathcal{C}_{u,z} \equiv (u,z) \cup \mathcal{P}_u$, and $\mathcal{C}_{v,z} \equiv (v,z) \cup \mathcal{P}_v$, and whose inclusion trees $T_{u,v}$, $T_{u,z}$, and $T_{v,z}$ are the subtrees of T induced by the clusters containing vertices of $G_{u,v}$, $G_{u,z}$, and $G_{v,z}$.

Lemma 14 *$C_{u,v}$, $C_{u,z}$, and $C_{v,z}$, are linearly-ordered outerclustered graphs.*

Fig. 27 shows graphs $C_{u,v}$, $C_{u,z}$, and $C_{v,z}$ when Condition 2 of Lemma 12 holds.

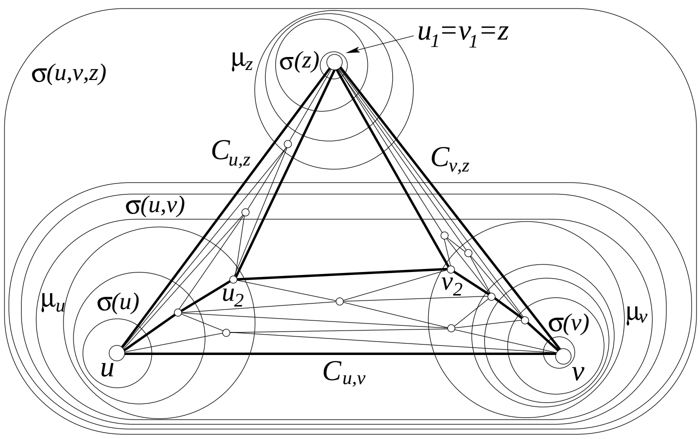


Figure 27: Graphs $C_{u,v}$, $C_{u,z}$, and $C_{v,z}$ when Condition 2 of Lemma 12 holds. Thick edges show paths $\mathcal{P}_u$, $\mathcal{P}_v$, and edges (u,v), (u,z), (v,z), and (u_2, v_2).

We are now ready to exhibit the main theorem of this section.

Theorem 2 *Let $C(G,T)$ be a maximal outerclustered graph. Then, for every triangular-convex-separated drawing $\Gamma(C_{f_o})$ of C_{f_o}, there exists an internally-convex-separated drawing $\Gamma(C)$ of C completing $\Gamma(C_{f_o})$.*

Proof: Let u, v, and z be the vertices incident to $f_o(G)$. Suppose that G has internal vertices, otherwise C_{f_o} and C are the same graph, and the statement trivially follows.

If $\sigma(u) = \sigma(v)$, then we claim that C is a linearly-ordered outerclustered graph. Refer to Fig. 28. Observe that C is a maximal outerclustered graph by hypothesis, hence G is triconnected and internally-triangulated.

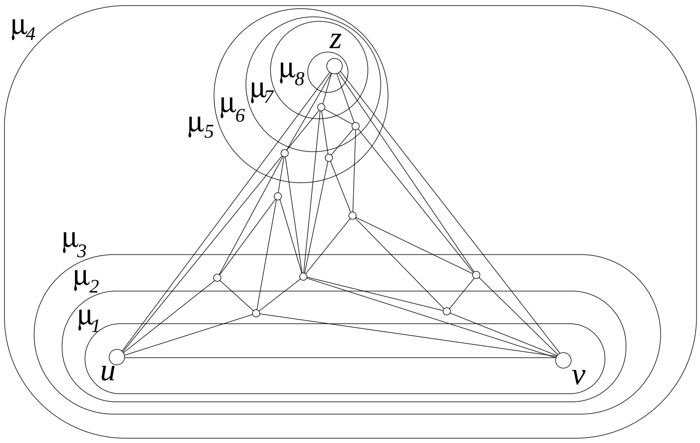


Figure 28: If $\sigma(u) = \sigma(v)$, then C is a linearly-ordered outerclustered graph

Define the following sequence of clusters $\Sigma = \mu_1, \mu_2, \dots, \mu_k$, where $\mu_1 = \sigma(u) = \sigma(v)$, μ_{i+1} is the parent of μ_i in T, for $i = 1, 2, \dots, h-2$, where μ_{h-1} is the biggest cluster containing u and v and not containing z, if any such a cluster exists, $\mu_h = \sigma(u, v, z)$, μ_{h+1} is the biggest cluster containing z and not containing u and v, if any such a cluster exists, μ_{i+1} is the only child of μ_i in T containing z, for $i = h+1, h+2, \dots, k-1$, and $\mu_k = \sigma(z)$. We claim that C is linearly-ordered according to Σ.

We prove that C satisfies Property LO1 of Definition 2. By construction, Σ includes all the clusters containing at least one out of u, v, and z, with the exception of the clusters that contain all of u, v, and z and that are ancestors of $\sigma(u, v, z)$; however, each cluster containing all of u, v, and z and ancestor of $\sigma(u, v, z)$ is different from $\sigma(x)$, for any vertex x of G, since x is also contained in $\sigma(u, v, z)$. Since, for each vertex x of G, $\sigma(x)$ is a cluster containing at least one out of u, v, and z, by definition of outerclustered graph, then $\sigma(x) = \mu_i$, for some $1 \leq i \leq k$.

We prove that C satisfies Property LO2. Path (u, v, z) and edge (u, z) are monotone paths delimiting $f_o(G)$.

We prove that C satisfies Property LO3. By construction, μ_{i+1} is the parent of μ_i, for $i = 1, 2, \dots, h-2$, and μ_{i+1} is a child of μ_i, for $i = h+1, h+2, \dots, k-1$. Hence, in order to prove that C satisfies Property LO3, it suffices to show that μ_h is the parent of μ_{h-1} and that μ_h is the parent of μ_{h+1}. By construction, μ_{h-1} (resp. μ_{h+1}) is the biggest cluster containing u and v and not containing z (resp. containing z and not containing u and v). Hence, the parent of μ_{h-1} (resp. of μ_{h+1}) is $\sigma(u, v, z)$, that by definition is μ_h.

Analogously, if $\sigma(u) = \sigma(z)$ or if $\sigma(v) = \sigma(z)$, C is a linearly-ordered outerclustered graph. By Lemma 4, a triangular-convex-separated drawing of C_{f_o} is also a convex-separated drawing of C_{f_o}, hence in such cases the theorem directly follows from Theorem 1.

Now suppose that $\sigma(u) \neq \sigma(v)$, $\sigma(u) \neq \sigma(z)$, and $\sigma(v) \neq \sigma(z)$. Suppose also that, if there exists a cluster containing exactly two vertices incident to $f_o(G)$, then such vertices are u and v. The cases in which such vertices are u and z, or v and z can be treated analogously.

By Lemma 12, either there exist three paths $\mathcal{P}_u = (u_1, u_2, \dots, u_U)$, $\mathcal{P}_v = (v_1, v_2, \dots, v_V)$, and $\mathcal{P}_z = (z_1, z_2, \dots, z_Z)$ satisfying Condition 1 of Lemma 12, or there exist two paths $\mathcal{P}_u = (u_1, u_2, \dots, u_U)$ and $\mathcal{P}_v = (v_1, v_2, \dots, v_V)$ satisfying Condition 2 of Lemma 12.

Suppose that Condition 1 of Lemma 12 holds.

By Lemma 13, $C_{u,v}(G_{u,v}, T_{u,v})$, $C_{u,z}(G_{u,z}, T_{u,z})$, and $C_{v,z}(G_{v,z}, T_{v,z})$ are linearly-ordered outerclustered graphs.

Consider a child $\sigma'(u_1)$ of $\sigma(u_1)$ such that: (i) if $\sigma(u_1)$ contains z and does not contain u and v, then $\sigma'(u_1)$ is the only child of $\sigma(u_1)$; (ii) if $\sigma(u_1) = \sigma(u, v, z)$, then $\sigma'(u_1)$ is the only child of $\sigma(u_1)$ containing z; (iii) if $\sigma(u_1)$ contains u and v and does not contain z, then $\sigma'(u_1)$ is the only child of $\sigma(u_1)$ containing u. Notice that in any case one of the two continuous lines obtained as intersection of the triangle representing (u, v, z) and $R(\sigma(u_1), \sigma'(u_1))$ lies on edge (u, z). Consider a point $p(z_1)$ of $int(R(\sigma(u_1), \sigma'(u_1)))$ arbitrarily close to edge (u, z). Place $u_1 = v_1 = z_1$ at $p(z_1)$.

Let $\sigma'(z_i)$ be the only child of $\sigma(z_i)$, for each $2 \leq i \leq Z - 1$. Consider a straight-line segment $\overline{p(z_1)z}$ and place z_i at any point of the segment $int(R(\sigma(z_i), \sigma'(z_i))) \cap \overline{p(z_1)z}$, for each $2 \leq l \leq Z - 1$.

Denote by $T(u, v, p(z_1))$ the triangle having u, v, and $p(z_1)$ as vertices. Denote also by $H(l(z, p(z_1)), u)$ (by $H(l(z, p(z_1)), v)$) the open half-plane delimited by the line through z and $p(z_1)$, and containing u (resp. containing v). Consider the only child $\sigma'(u_2)$ of $\sigma(u_2)$. Notice that one of the two continuous lines obtained as intersection of the triangle representing (u, v, z) and $R(\sigma(u_2), \sigma'(u_2))$ lies on edge (u, z). Consider a point $p(u_2)$ in $H(l(z, p(z_1)), u) \cap int(T(u, v, p(z_1))) \cap int(R(\sigma(u_2), \sigma'(u_2)))$ arbitrarily close to edge (u, z). Place u_2 at $p(u_2)$.

Let $\sigma'(u_i)$ be the only child of $\sigma(u_i)$, for each $3 \leq i \leq U - 1$. Consider a straight-line segment $\overline{p(u_2)u}$ and place u_i at any point of the segment $int(R(\sigma(u_i), \sigma'(u_i)))$ $\cap\, \overline{p(u_2)u}$, for each $3 \leq i \leq U - 1$.

Denote by $T(p(u_2), v, p(z_1))$ the triangle having $p(u_2)$, v, and $p(z_1)$ as vertices. Denote also by $H(l(u, p(u_2)), v)$ the open half-plane delimited by the line through u and $p(u_2)$, and containing v. Consider the only child $\sigma'(v_2)$ of $\sigma(v_2)$. Consider any point $p(v_2)$ in $H(l(z, p(z_1)), v) \cap H(l(u, p(u_2)), v) \cap int(T(p(u_2), v, p(z_1))) \cap int(R(\sigma(v_2), \sigma'(v_2)))$. Observe that, since $p(z_1)$ and $p(u_2)$ are arbitrarily close to edge (u, z), both half-planes $H(l(z, p(z_1)), v)$ and $H(l(u, p(u_2)), v)$ entirely contain triangle $T(u, v, z)$, except for an arbitrarily small strip close to edge (u, z). This guarantees that $H(l(z, p(z_1)), v) \cap H(l(u, p(u_2)), v) \cap int(T(p(u_2), v, p(z_1))) \cap int(R(\sigma(v_2), \sigma'(v_2)))$ is a convex non-empty region. Then, place v_2 at $p(v_2)$.

Let $\sigma'(v_i)$ be the only child of $\sigma(v_i)$, for each $3 \leq i \leq V - 1$. Consider a straight-line segment $\overline{p(v_2)v}$ and place v_i at any point of the segment $int(R(\sigma(v_i), \sigma'(v_i))) \cap \overline{p(v_2)v}$, for each $3 \leq i \leq V - 1$. See Fig. 29. Straightforward modifications make the described algorithm work also for the cases in which $U = 2$ and/or $V = 2$.

Denote by $\Gamma(\mathcal{C}_{u,v}, T_{u,v})$, by $\Gamma(\mathcal{C}_{u,z}, T_{u,z})$, and by $\Gamma(\mathcal{C}_{v,z}, T_{v,z})$ the constructed drawings of $(\mathcal{C}_{u,v}, T_{u,v})$, of $(\mathcal{C}_{u,z}, T_{u,z})$, and of $(\mathcal{C}_{v,z}, T_{v,z})$, respectively.

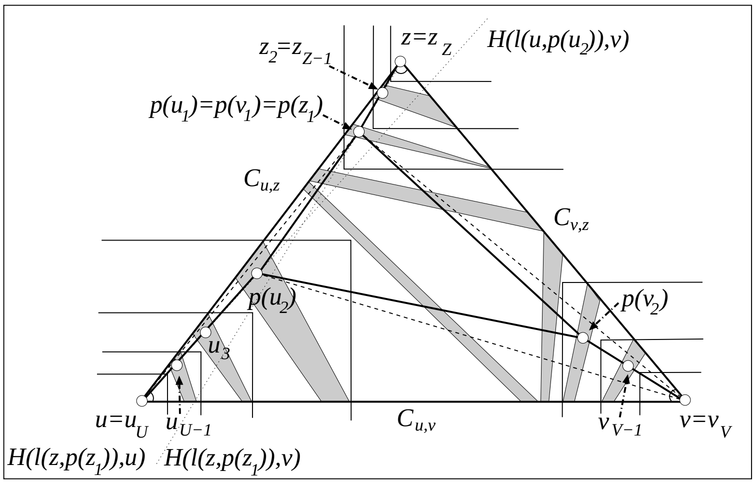


Figure 29: Construction of drawings $\Gamma(\mathcal{C}_{u,v}, T_{u,v})$, $\Gamma(\mathcal{C}_{u,z}, T_{u,z})$, and $\Gamma(\mathcal{C}_{v,z}, T_{v,z})$ when Condition 1 of Lemma 12 holds.

Lemma 15 *$\Gamma(\mathcal{C}_{u,v}, T_{u,v})$, $\Gamma(\mathcal{C}_{u,z}, T_{u,z})$, and $\Gamma(\mathcal{C}_{v,z}, T_{v,z})$ are convex-separated drawings of the outer faces of $C_{u,v}(G_{u,v}, T_{u,v})$, $C_{u,z}(G_{u,z}, T_{u,z})$, and $C_{v,z}(G_{v,z}, T_{v,z})$, respectively.*

Proof: We prove the statement for $\Gamma(\mathcal{C}_{u,v}, T_{u,v})$, the proof for $\Gamma(\mathcal{C}_{u,z}, T_{u,z})$, and $\Gamma(\mathcal{C}_{v,z}, T_{v,z})$ being analogous. Denote by P, $P_{u,v}$, $P_{u,z}$, and $P_{v,z}$ the polygons representing cycles $\mathcal{C}$, $\mathcal{C}_{u,v}$, $\mathcal{C}_{u,z}$, and $\mathcal{C}_{v,z}$, respectively. The drawing is straight-line and rectangular by construction. The absence of edge crossings easily descends from the construction. The absence of region-region crossings descends from the fact that no cluster is drawn by the algorithm.

We prove that $\Gamma(\mathcal{C}_{u,v}, T_{u,v})$ has no edge-region crossings. Suppose, for a contradiction, that there is an edge-region crossing between an edge e and a cluster ν. If both end-vertices of e belong to ν then, by the convexity of ν, e would be internal to ν; if one end-vertex of e belongs to ν then, by the convexity of ν, e would cross ν exactly once; hence, it can be assumed that both end-vertices of e do not belong to ν. Any cluster containing both u and v contains all the vertices of $G_{u,v}$, hence it contains all the drawing of $\mathcal{C}_{u,v}$ and does not cross e. Hence, it can be assumed that ν contains u and does not contain v, or vice versa. Suppose that ν contains u and does not contain v, the other case being analogous. Consider the parent μ of ν in $T_{u,v}$. Such a parent exists otherwise ν would be the root of T, contradicting the fact that ν does not contain v. By definition of triangular-convex-separated drawing, there exists a convex region $R(\mu, \nu)$ with the properties described in Definition 4; such a region separates ν from the rest of the drawing, thus avoiding an edge-region crossing between e and ν. More precisely, by definition of outerclustered graph, ν has exactly two incident edges $e_1(\nu)$ and $e_2(\nu)$. Denote by $u(e_1(\nu))$ and $u(e_2(\nu))$ the endvertices of $e_1(\nu)$ and $e_2(\nu)$ belonging to ν. Denote by $p(l_1)$ and $p(l_2)$ the endpoints of $l_1(\mu, \nu)$ and $l_2(\mu, \nu)$ closer to $u(e_1(\nu))$ and $u(e_2(\nu))$, respectively. Then, segment $\overline{p(l_1)p(l_2)}$ splits P in two convex polygons P_1 and P_2, where P_1 contains all and only the vertices in ν and P_2 contains all the vertices not belonging to ν. By convexity, e is internal to P_2, while the part of ν inside P is internal to P_1. Hence, e does not cross ν.

We prove that $\Gamma(\mathcal{C}_{u,v}, T_{u,v})$ satisfies Property CS1 of Definition 3. The angles $\widehat{u_2uv}$ and $\widehat{v_2vu}$ incident to u and v inside $P_{u,v}$ are strictly less than 180°, since they are respectively less than angles $\widehat{zuv}$ and $\widehat{zvu}$, that are angles of P, which is a triangle. By construction, v_2

is contained inside triangle $T(p(u_2), v, p(z_1))$, hence $\widehat{u_2v_2v}$ is the angle of a triangle having u_2, v_2, and v as vertices, hence it is less than 180°. Finally, angle $\widehat{uu_2v}$ is less than 180°, since by construction v_2 is placed in the half-plane $H(l(u, p(u_2)), v)$ delimited by the line through u and $p(u_2)$, and containing v.

We prove that $\Gamma(\mathcal{C}_{u,v}, T_{u,v})$ satisfies Property CS2. Observe that $\sigma(u)$ and $\sigma(v)$ are the first and the last cluster in Σ, respectively, and that the angles $\widehat{u_2uv}$ and $\widehat{v_2vu}$ incident to u and v inside $P_{u,v}$ are strictly less than 180°, as proved above.

We prove that $\Gamma(\mathcal{C}_{u,v}, T_{u,v})$ satisfies Property CS3. The existence of regions $R(\mu, \nu)$ inside $P_{u,v}$ descends from the existence of regions $R(\mu, \nu)$ inside P, where ν is any child of μ in $T_{u,v}$. Namely, the drawn edges cut each convex region $R(\mu, \nu)$ into two or three convex regions, that satisfy the properties that have to be satisfied by $R(\mu, \nu)$ inside $P_{u,v}$, as can be easily deduced from the fact that the same properties are satisfied by $R(\mu, \nu)$ inside P. □

Graphs $C_{u,v}$, $C_{u,z}$, and $C_{v,z}$ are, in general, not triconnected; namely, there could exist chords: (i) in $\mathcal{C}_{u,v}$ between any vertex in $\mathcal{P}_u \setminus \{u_1\}$ and any vertex in $\mathcal{P}_v \setminus \{v_1\}$; (ii) in $\mathcal{C}_{u,z}$ between vertex u_1 and any vertex in $\mathcal{P}_u \setminus \{u_1, u_2\}$, and between any vertex in $\mathcal{P}_u \setminus \{u_1\}$ and any vertex in $\mathcal{P}_z \setminus \{z_1\}$; (iii) in $\mathcal{C}_{v,z}$ between vertex v_1 and any vertex in $\mathcal{P}_v \setminus \{v_1, v_2\}$, and between any vertex in $\mathcal{P}_v \setminus \{v_1\}$ and any vertex in $\mathcal{P}_z \setminus \{z_1\}$. By Lemma 2, each of such chords splits a linearly-ordered outerclustered graph into two smaller linearly-ordered outerclustered graphs. Further, by construction the endvertices of each of such chords are not collinear with any other vertex of the cycle. Hence, by Lemma 3, inserting the chords as straight-line segments into drawings $\Gamma(\mathcal{C}_{u,v}, T_{u,v})$, $\Gamma(\mathcal{C}_{u,z}, T_{u,z})$, and $\Gamma(\mathcal{C}_{v,z}, T_{v,z})$, that are convex-separated by Lemma 15, splits them into convex-separated drawings. When all the chords have been added, the underlying graphs of the resulting clustered graphs are all triconnected and internally-triangulated. Hence, Theorem 2 applies and an internally-convex-separated drawing of each of such linearly-ordered outerclustered graphs can be constructed inside the corresponding outer face, thus obtaining an internally-convex-separated drawing of C.

Now suppose that Condition 2 of Lemma 12 holds.

By Lemma 14, $C_{u,v}(G_{u,v}, T_{u,v})$, $C_{u,z}(G_{u,z}, T_{u,z})$, and $C_{v,z}(G_{v,z}, T_{v,z})$ are linearly-ordered outerclustered graphs.

Consider the only child $\sigma'(u_i)$ of $\sigma(u_i)$, for each $2 \leq i \leq U - 1$. Notice that one of the two continuous lines obtained as intersection of the triangle representing (u, v, z) and $R(\sigma(u_2), \sigma'(u_2))$ lies on edge (u, z). Consider a point $p(u_2)$ of $int(R(\sigma(u_2), \sigma'(u_2)))$ arbitrarily close to edge (u, z). Place u_2 at $p(u_2)$. Consider a straight-line segment $\overline{p(u_2)u}$ and place u_i at any point of the segment $int(R(\sigma(u_i), \sigma'(u_i))) \cap \overline{p(u_2)u}$, for each $3 \leq i \leq U - 1$.

Denote by $T(p(u_2), v, z))$ the triangle having $p(u_2)$, v, and z as vertices. Denote also by $H(l(u, p(u_2)), v)$ the open half-plane delimited by the line through u and $p(u_2)$, and containing v.

Let $\sigma'(v_i)$ be the only child of $\sigma(v_i)$, for each $2 \leq i \leq V - 1$. Consider any point $p(v_2)$ in $H(l(u, p(u_2)), v) \cap int(T(p(u_2), v, z)) \cap int(R(\sigma(v_2), \sigma'(v_2)))$ and place v_2 at $p(v_2)$. Consider a straight-line segment $\overline{p(v_2)v}$ and place v_i at any point of the segment $int(R(\sigma(v_i), \sigma'(v_i))) \cap \overline{p(v_2)v}$, for each $3 \leq i \leq V - 1$. See Fig. 30. Straightforward modifications make the described algorithm work also for the cases in which $U = 2$ or $V = 2$ (notice that $U = 2$ and $V = 2$ do not hold simultaneously when Condition 2 of Lemma 12 holds, otherwise

G would not have internal vertices).

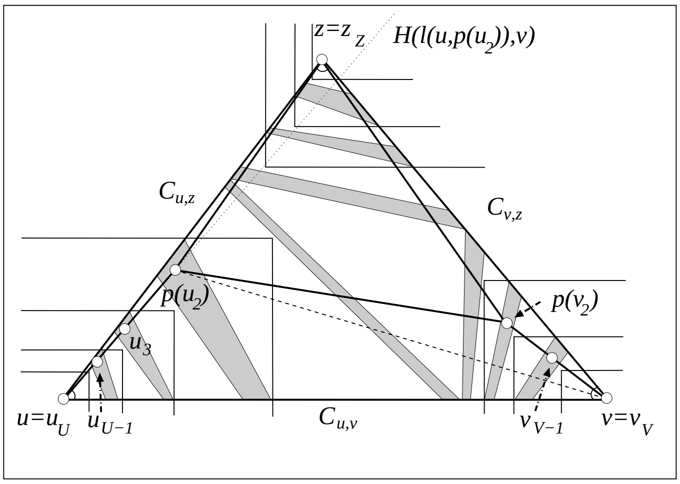


Figure 30: Construction of drawings $\Gamma(\mathcal{C}_{u,v}, T_{u,v})$, $\Gamma(\mathcal{C}_{u,z}, T_{u,z})$, and $\Gamma(\mathcal{C}_{v,z}, T_{v,z})$ when Condition 2 of Lemma 12 holds.

Denote by $\Gamma(\mathcal{C}_{u,v}, T_{u,v})$, by $\Gamma(\mathcal{C}_{u,z}, T_{u,z})$, and by $\Gamma(\mathcal{C}_{v,z}, T_{v,z})$ the constructed drawings of $(\mathcal{C}_{u,v}, T_{u,v})$, of $(\mathcal{C}_{u,z}, T_{u,z})$, and of $(\mathcal{C}_{v,z}, T_{v,z})$, respectively.

Lemma 16 *$\Gamma(\mathcal{C}_{u,v}, T_{u,v})$, $\Gamma(\mathcal{C}_{u,z}, T_{u,z})$, and $\Gamma(\mathcal{C}_{v,z}, T_{v,z})$ are convex-separated drawings of the outer faces of $C_{u,v}(G_{u,v}, T_{u,v})$, $C_{u,z}(G_{u,z}, T_{u,z})$, and $C_{v,z}(G_{v,z}, T_{v,z})$, respectively.*

Graphs $C_{u,v}$, $C_{u,z}$, and $C_{v,z}$ are, in general, not triconnected; namely, there could exist chords: (i) in $\mathcal{C}_{u,v}$ between any vertex in $\mathcal{P}_u \setminus \{u_1\}$ and any vertex in $\mathcal{P}_v \setminus \{v_1\}$; (ii) in $\mathcal{C}_{u,z}$ between vertex u_1 and any vertex in $\mathcal{P}_u \setminus \{u_1\}$; (iii) in $\mathcal{C}_{v,z}$ between vertex v_1 and any vertex in $\mathcal{P}_v \setminus \{v_1\}$. By Lemma 2, each of such chords splits a linearly-ordered outerclustered graph into two smaller linearly-ordered outerclustered graphs. Further, by construction the endvertices of each of such chords are not collinear with any other vertex of the cycle. Hence, by Lemma 3, inserting the chords as straight-line segments into drawings $\Gamma(\mathcal{C}_{u,v}, T_{u,v})$, $\Gamma(\mathcal{C}_{u,z}, T_{u,z})$, and $\Gamma(\mathcal{C}_{v,z}, T_{v,z})$, that are convex-separated by Lemma 16, splits them into convex-separated drawings. When all chords have been added, the underlying graphs of the resulting clustered graphs are all triconnected and internally-triangulated. Hence, Theorem 2 applies and an internally-convex-separated drawing of each of such linearly-ordered outerclustered graphs can be constructed inside the corresponding outer face, thus obtaining an internally-convex-separated drawing of C. □

5 Drawing Clustered Graphs

In this section we prove the following theorem:

Theorem 3 *Let $C(G,T)$ be a maximal c-planar clustered graph. Then, for every triangular-convex-separated drawing $\Gamma(C_{f_o})$ of C_{f_o}, there exists an internally-convex-separated drawing $\Gamma(C)$ of C completing $\Gamma(C_{f_o})$.*

The proof is by induction on the number of vertices of G plus the number of clusters in T. In the base case, C is an outerclustered graph and the statement follows from Theorem 2. Consider any maximal clustered graph $C(G,T)$.

Case 1: There exists a minimal cluster μ containing exactly one vertex v internal to G and containing no vertex incident to $f_o(G)$. Remove μ from T obtaining a clustered graph $C'(G,T')$. Observe that C_{f_o} and C'_{f_o} are the same graph. The number of vertices plus the number of clusters in C' is one less than in C. Hence, the inductive hypothesis applies and there exists an internally-convex-separated drawing $\Gamma(C')$ of C' completing an arbitrary triangular-convex-separated drawing $\Gamma(C_{f_o})$ of C_{f_o}. In $\Gamma(C')$ a small disk D can be drawn centered at v, not intersecting the border of any cluster, not containing any vertex of G different from v, and intersecting only the edges incident to v. For each edge e_i incident to v, choose two points p_i^1 and p_i^2 inside D, where p_i^1 is closer to v than p_i^2. Insert a drawing of μ in $\Gamma(C')$ as a rectangle containing v and lying inside the polygon $(p_1^1, p_2^1, \dots, p_k^1, p_1^1)$, thus obtaining a drawing $\Gamma(C)$. Fig. 31 illustrates Case 1 of the algorithm for drawing clustered graphs.

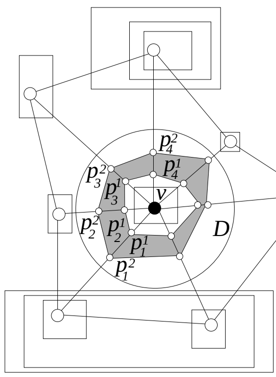


Figure 31: Illustration for Case 1 of the algorithm for drawing clustered graphs.

Lemma 17 $\Gamma(C)$ *is an internally-convex-separated drawing of C.*

Proof: $\Gamma(C)$ has no edge crossing, by induction. $\Gamma(C)$ has no edge-region crossing since any cluster different from μ has no crossing with any edge by induction, and since μ does not intersect any edge not incident to v, because it completely lies inside D. $\Gamma(C)$ has no region-region crossing since any two clusters different from μ have no crossing by induction, and since μ does not intersect the border of any cluster, because it completely lies inside D. The drawing is straight-line and rectangular by construction. Further, every internal face of G not incident to v is triangular-convex-separated by induction. Finally, for each two edges e_i and e_{i+1} consecutively incident to v denote by f_i the face of G incident to edges e_i and e_{i+1}, and denote by $R(\mu, i)$ the quadrilateral having p_i^1, p_i^2, p_{i+1}^1, and p_{i+1}^2 as vertices. Then, for each face f_i, region $R(\nu,\mu) \equiv R(\mu, i)$, where ν is the parent of μ in T, satisfies Property TCS1 of a triangular-convex-separated drawing, due to the fact that such a region is completely contained inside D, and that ν completely contains $R(\nu,\mu)$. □

Case 2: There exists a separating 3-cycle (u', v', z') in G. Let $C^1(G^1, T^1)$ ($C^2(G^2, T^2)$) be the clustered graph defined as follows. G^1 (resp. G^2) is the subgraph of G induced by u', v', z', and by the vertices outside (u', v', z') (resp. by u', v', z', and by the vertices inside (u', v', z')). T^1 (resp. T^2) is the subtree of T whose clusters contain vertices of G^1 (resp. of G^2). Observe that C_{f_o} and $C^1_{f_o}$ are the same graph. Since (u', v', z') is a separating 3-cycle, the number of vertices plus the number of clusters in each of C^1 and C^2 is strictly less than in C. Hence, the inductive hypothesis applies and there exists an internally-convex-separated drawing $\Gamma(C^1)$ of C^1 completing an arbitrary triangular-convex-separated drawing $\Gamma(C_{f_o})$ of C_{f_o}. Cycle (u', v', z') is a face f of G^1. Then, the drawing $\Gamma(C_f)$ of C_f in $\Gamma(C^1)$ is a triangular-convex-separated drawing. Observe that C_f and $C^2_{f_o}$ are the same graph. Hence, the inductive hypothesis applies again and an internally-convex-separated drawing $\Gamma(C^2)$ can be constructed completing $\Gamma(C^2_{f_o})$. Plugging $\Gamma(C^2)$ in $\Gamma(C^1)$ provides a drawing $\Gamma(C)$ of C.

Lemma 18 $\Gamma(C)$ *is a internally-convex-separated drawing of* C.

Proof: $\Gamma(C)$ has no edge crossing. Namely, any edge belonging to G^1 (resp. to G^2) does not cross any edge belonging to G^1 (resp. to G^2) by induction. Further, any edge belonging to G^1 and not belonging to G^2 does not cross any edge belonging to G^2 and not belonging to G^1 since such edges are separated by cycle (u', v', z'). $\Gamma(C)$ has no edge-region crossing. Namely, any edge belonging to G^1 (resp. to G^2) does not cross the border of any cluster of T^1 (resp. of T^2) by induction. Further, any edge belonging to G^1 (resp. to G_2) and not belonging to G^2 (resp. to G^1) does not cross the border of any cluster belonging to T^2 (resp. to T^1) and not belonging to T^1 (resp. to T^2), since such an edge and such a cluster are separated by cycle (u', v', z'). $\Gamma(C)$ has no region-region crossing. Namely, the border of any cluster belonging to T^1 (resp. to T^2) does not cross the border of any cluster belonging to T^1 (resp. to T^2) by induction. Further, the border of any cluster belonging to T^1 (resp. to T^2) and not belonging to T^2 (resp. to T^1), does not cross the border of any cluster belonging to T^2 (resp. to T^1) and not belonging to T^1 (resp. to T^2), since such clusters are separated by cycle (u', v', z'). $\Gamma(C)$ is straight-line and rectangular by construction. Further, the drawing of any internal face f of G is triangular-convex-separated since it is triangular-convex-separated in $\Gamma(C^1)$ (if f is also a face of G^1) or in $\Gamma(C^2)$ (if f is also a face of G^2). □

Case 3: There exists no separating 3-cycle, and there exist two adjacent vertices u' *and* v' *such that* $\sigma(u') = \sigma(v')$ *and such that they are not both external.* Suppose that G contains two adjacent vertices u' and v' such that $\sigma(u') = \sigma(v')$ and such that u' is internal, and suppose that there exists no separating 3-cycle in G. Since G is maximal, u' and v' have exactly two common neighbors z'_1 and z'_2. Contract edge (u', v') to a vertex w', that is, replace vertices u' and v' with a vertex w' connected to all the vertices u' and v' are connected to. Vertex w' belongs to $\sigma(u')$ and to all the ancestors of $\sigma(u')$ in T'. The resulting clustered graph $C'(G', T')$ is easily shown to be a maximal c-planar clustered graph. In particular, the absence of separating 3-cycles in G guarantees that G' is simple and maximal. Observe that C_{f_o} and C'_{f_o} are the same graph. Hence, the inductive hypothesis applies and there exists an internally-convex-separated drawing $\Gamma(C')$ of C' completing an arbitrary triangular-convex-separated drawing $\Gamma(C_{f_o})$ of C_{f_o}. Then consider a small disk D centered at w' and consider any line l from w' to an interior point of the segment between z'_1 and z'_2. Replace w' with u' and v' so that such vertices lie on l and inside D. Connect u' and v' to their neighbors, obtaining a drawing $\Gamma(C)$ of C. Fig. 32 illustrates Case 3 of the algorithm for drawing clustered graphs.

Lemma 19 $\Gamma(C)$ *is an internally-convex-separated drawing of* C.

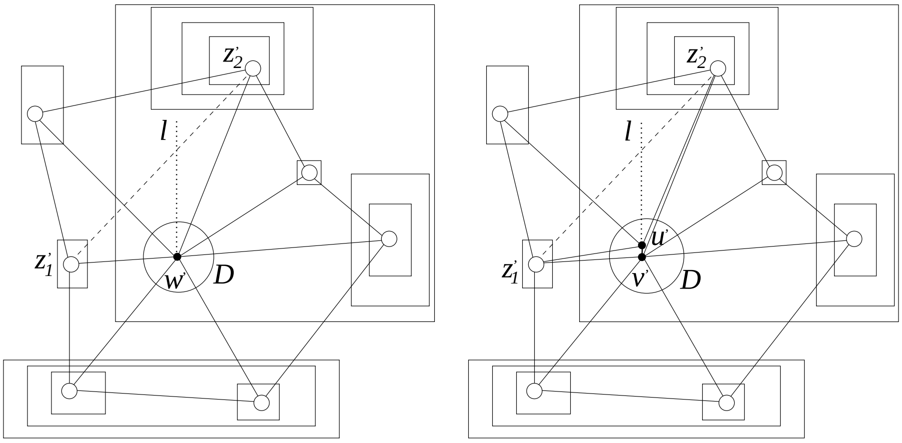


Figure 32: Illustration for Case 3 of the algorithm for drawing clustered graphs.

Proof: The existence in a straight-line drawing of a disk D centered at any vertex w' such that moving w' to any point inside D leaves the straight-line drawing planar has been first proved in [10] taking into account the set of points from which all the neighbors of w' are *visible*, i.e., straight-line segments can be drawn without causing crossings. This has been exploited in [10] to argue that, in a graph G with no separating 3-cycle, an edge (u', v') that has been contracted to a single vertex w' (obtaining a graph G') can suitably replace w' so that the resulting straight-line drawing of G is planar. Here the continuity arguments used in [10] to prove the existence of D are still valid; however, the visibility between any point p of D and any neighbor z' of w' means that it is possible to draw a straight-line segment from p to z' not crossing any edge of G and not crossing twice the border of the same cluster; further, D has to be so small that it does not intersect the border of any cluster.

Then, the placement of u' and v' guarantees that $\Gamma(C)$ has no edge crossing and no edge-region crossing. Further, $\Gamma(C)$ has no region-region crossing, by induction. $\Gamma(C)$ is a straight-line rectangular drawing, by construction. Finally, for each internal face f of G not incident to u' and v' regions $R(\mu, \nu)$ can be drawn as in $\Gamma(C')$, since f has the same drawing in $\Gamma(C)$ and in $\Gamma(C')$; for each internal face f of G incident to u' and not to v', or viceversa, regions $R(\mu, \nu)$ can be drawn similarly to $\Gamma(C')$, since the drawings of f in $\Gamma(C)$ and in $\Gamma(C')$ (in C', face f is incident to vertex w', that replaces the one out of u' and v' that is incident to f in C) differ for an arbitrary small displacement of an incident vertex; faces (u', v', z'_1) and (u', v', z'_2) are so thin that no vertex of any rectangle representing a cluster lies inside such faces, hence regions $R(\mu, \nu)$ can easily be drawn. □

It remains to prove that the case in which C is an outerclustered graph is the base case.

Lemma 20 *Suppose that none of Cases 1, 2, and 3 applies. Then C is an outerclustered graph.*

Proof: Suppose, for a contradiction, that none of Cases 1, 2, and 3 applies, and that C is not an outerclustered graph. By the maximality of G, $f_o(G)$ is delimited by a 3-cycle. By the c-planarity of C, each cluster μ that contains a vertex incident to $f_o(G)$ and that does not contains all the vertices incident to $f_o(G)$ intersects $f_o(G)$ exactly twice. Namely, since the border of μ is a simple closed curve, it intersects $f_o(G)$ an even number of times.

However, since $f_o(G)$ has three edges, the border of μ does not intersect $f_o(G)$ more than three times. This proves Property O2 of Definition 1.

Suppose that C contains a cluster μ not containing any vertex incident to $f_o(G)$. Then, C contains a minimal cluster μ' not containing any vertex incident to $f_o(G)$, namely $\mu' = \mu$ if μ is minimal, and μ' is any minimal cluster descendant of μ, if μ is not minimal. If μ' contains exactly one vertex v, then μ' is a minimal cluster containing only v, and Case 1 would apply. If μ' contains more than one vertex, then by the c-connectivity of C, there exists at least one edge (u, v) such that $\sigma(u) = \sigma(v) = \mu'$. If (u, v) is an edge of a separating 3-cycle, then Case 2 would apply. Otherwise, Case 3 would apply. This proves Property O1.

Suppose that C contains an edge (u, v) such that $\sigma(u) = \sigma(v)$ and suppose that at least one out of u and v is an internal vertex of G. If edge (u, v) belongs to a separating 3-cycle, then Case 2 would apply. Otherwise, Case 3 would apply. This proves Property O3.

□

6 Conclusions

In this paper we have shown that every c-planar clustered graph admits a c-planar straight-line rectangular drawing. Actually, the algorithms we proposed do not exploit at all the fact that clusters are drawn as rectangles. The only property that must be satisfied by each region representing a cluster for the algorithm to work is that an edge incident to the cluster should cross its border exactly once. Hence, the algorithm we proposed can be modified in order to construct a c-planar straight-line drawing of a given clustered graph for an arbitrary assignment of convex shapes to the clusters (actually, *star-shaped polygons* are more in general feasible, i.e., polygons that have a set of points, called *kernel*, from which it is possible to draw edges towards all the vertices of the polygon without crossing its sides).

The algorithm we described in this paper uses real coordinates, hence it requires exponential area to be implemented in a system with a finite resolution rule. However, this drawback is unavoidable, since it has been proved by Feng *et al.* [12] that there exist clustered graphs requiring exponential area in any straight-line drawing in which clusters are represented by convex regions. We believe worth of interest the problem of determining whether clustered graphs whose hierarchy is *flat*, i.e., all clusters different from the root do not contain smaller clusters, admit straight-line convex drawings and straight-line rectangular drawings in polynomial area.

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