On Agreement Problems with Gossip Algorithms in absence of common reference frames

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International Conference on Robotics and Automation,
Anchorage, Alaska, USA - May 2010

Outline

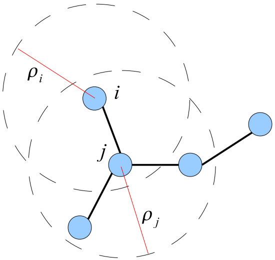
Multi-Agent System Modeling

A multi-agent system (MAS) is a system composed of multiple interacting intelligent agents. Agents can be of any kind, e.g., software agents, robots.

- The network topology can be described through a time-varying proximity graph $\mathcal{G}(t) = (V, E(t))$.
- An interaction between two agents $\{i, j\}$ occurs iff:

$$\|p_i(t) - p_j(t)\| \leq \min\{\rho_i, \rho_j\},$$

with $p_i(t) \in \mathbb{R}^d$ and $\rho_i \in \mathbb{R}$ respectively the agent position and its sensing radius.



Motivations

Multi-Agent Systems represent a valid framework to develop decentralized motion coordination algorithms.

One (or more) of the following assumptions are usually made:

1. Agents share a common reference frame.
2. Agents can access absolute position information, (e.g., GPS).
3. Agents have a common (absolute) attitude reference, (e.g., compass).

$$\Downarrow$$

A way to retrieve global information by exploiting only local measurements would significantly advance the feasibility of multi-agent systems.

Objective

To design decentralized approaches to locally retrieve global information usually not available to the agents.

In particular the following problems have been addressed:

1. Agreement on a common point in order to obtain a common landmark.
2. Agreement on a common reference frame in order to obtain a common position system

Under the following assumptions:

1. No hardware for global position system is available.
2. Each agent has it own reference frame unknown to the others.

Framework Description (I)

The following further assumptions are made on the MAS:

- Each agent is characterized by a position in a 2D space.
- The network is described by a undirected switching graph.
- Sensing range is limited by a maximum sensing radius ρ.
- Communications are asynchronous , *gossip* like.
- Each agent can identify its neighbors.
- Each agent can sense the distance between itself and its neighbors.
- Each agent can sense the direction in which it sees its neighbors with respect to its local reference frame.

Framework Description (II)

In the proposed framework a gossip algorithm is defined as a triplet $\{\mathcal{S}, \mathcal{R}, \mathfrak{e}\}$ where:

- $\mathcal{S} = \{s_1, \ldots, s_n\}$ is a set containing the local estimate s_i of each agent i in the network.
- $\mathcal{R}$ is a local interaction rule that given edge e_{ij} and the states of agents i, j $\mathcal{R} : (s_i, s_j) \Rightarrow (\hat{s}_i, \hat{s}_j)$.
- $\mathfrak{e}$ is a edge selection process that specifies which edge $e_{ij} \in E(t)$ is selected at time t.

Outline

Agreement on a Common Point

Our first objective is to make the local estimate of each agent converge to a common value by applying an iterative algorithm so that:

$$\forall i, \quad \lim_{t\to\infty} s_{gi}(t) = R_i s_i(t) + p_i = \frac{1}{n}\sum_{i=1}^{n}\Big(R_i s_i(0) + p_i\Big). \qquad (1)$$

Note that, for each agent i the local estimate s_i can be expressed with respect to a global reference frame as follows:

$$s_{gi} = R_i s_i + p_i. \qquad (2)$$

where R_i is a rotation matrix to move from the reference frame $\mathcal{O}_i$ of the agent i to the global $\mathcal{O}$.

Interaction Rule $\mathcal{R}$ (I)

If two agents (i, j) are selected at time t, their local estimate is updated as follows:

$$\begin{aligned} s_i(t+1) = & \quad \Delta(t) \cdot \hat{c}_{ij} + \Delta^{\perp}(t) \cdot \hat{c}_{ij}^{\perp}, \\ s_j(t+1) = & \quad \Delta(t) \cdot \hat{c}_{ji} + \Delta^{\perp}(t) \cdot \hat{c}_{ji}^{\perp}. \end{aligned} \tag{3}$$

where:

$$\begin{aligned} \Delta(t) &= \frac{d_{ij} - s_j(t)^T \hat{c}_{ji} + s_i(t)^T \hat{c}_{ij}}{2}, \\ \Delta^{\perp}(t) &= \frac{s_i(t)^T \hat{c}_{ij}^{\perp} - s_j(t)^T \hat{c}_{ji}^{\perp}}{2}, \end{aligned} \tag{4}$$

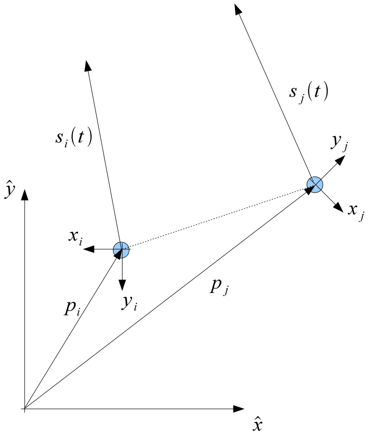
Interaction Rule $\mathcal{R}$ (II)

If two agents (i, j) are selected at time t, their local estimate is updated as follows:

$$s_i(t+1) = \Delta(t) \cdot \hat{c}_{ij} + \Delta^{\perp}(t) \cdot \hat{c}_{ij}^{\perp},$$
$$s_j(t+1) = \Delta(t) \cdot \hat{c}_{ji} + \Delta^{\perp}(t) \cdot \hat{c}_{ji}^{\perp}.$$

where:

$$\Delta(t) = \frac{d_{ij} - s_j(t)^T \hat{c}_{ji} + s_i(t)^T \hat{c}_{ij}}{2},$$
$$\Delta^{\perp}(t) = \frac{s_i(t)^T \hat{c}_{ij}^{\perp} - s_j(t)^T \hat{c}_{ji}^{\perp}}{2},$$



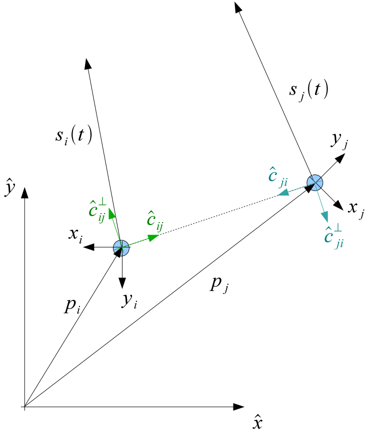
Interaction Rule $\mathcal{R}$ (III)

If two agents (i, j) are selected at time t, their local estimate is updated as follows:

$$s_i(t+1) = \Delta(t) \cdot \hat{c}_{ij} + \Delta^{\perp}(t) \cdot \hat{c}_{ij}^{\perp},$$
$$s_j(t+1) = \Delta(t) \cdot \hat{c}_{ji} + \Delta^{\perp}(t) \cdot \hat{c}_{ji}^{\perp}.$$

where:

$$\Delta(t) = \frac{d_{ij} - s_j(t)^T \hat{c}_{ji} + s_i(t)^T \hat{c}_{ij}}{2},$$
$$\Delta^{\perp}(t) = \frac{s_i(t)^T \hat{c}_{ij}^{\perp} - s_j(t)^T \hat{c}_{ji}^{\perp}}{2},$$



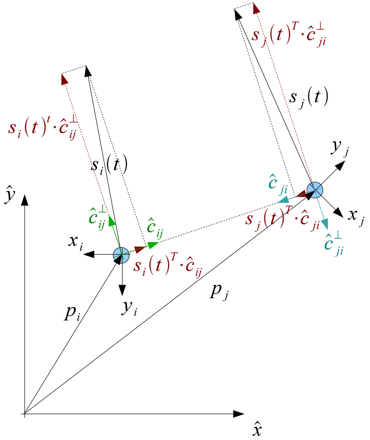
Interaction Rule $\mathcal{R}$ (IV)

If two agents (i, j) are selected at time t, their local estimate is updated as follows:

$$s_i(t+1) = \Delta(t) \cdot \hat{c}_{ij} + \Delta^{\perp}(t) \cdot \hat{c}_{ij}^{\perp},$$
$$s_j(t+1) = \Delta(t) \cdot \hat{c}_{ji} + \Delta^{\perp}(t) \cdot \hat{c}_{ji}^{\perp}.$$

where:

$$\Delta(t) = \frac{d_{ij} - s_j(t)^T \hat{c}_{ji} + s_i(t)^T \hat{c}_{ij}}{2},$$
$$\Delta^{\perp}(t) = \frac{s_i(t)^T \hat{c}_{ij}^{\perp} - s_j(t)^T \hat{c}_{ji}^{\perp}}{2},$$



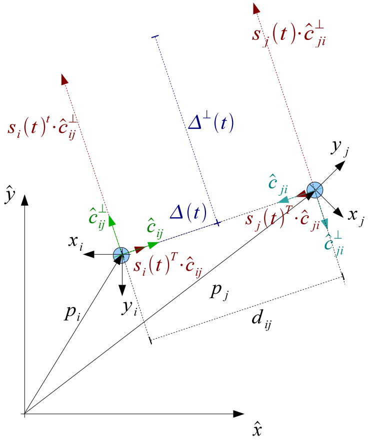
Interaction Rule $\mathcal{R}$ (V)

If two agents (i, j) are selected at time t, their local estimate is updated as follows:

$$s_i(t+1) = \Delta(t) \cdot \hat{c}_{ij} + \Delta^{\perp}(t) \cdot \hat{c}_{ij}^{\perp},$$
$$s_j(t+1) = \Delta(t) \cdot \hat{c}_{ji} + \Delta^{\perp}(t) \cdot \hat{c}_{ji}^{\perp}.$$

where:

$$\Delta(t) = \frac{d_{ij} - s_j(t)^T \hat{c}_{ji} + s_i(t)^T \hat{c}_{ij}}{2},$$
$$\Delta^{\perp}(t) = \frac{s_i(t)^T \hat{c}_{ij}^{\perp} - s_j(t)^T \hat{c}_{ji}^{\perp}}{2},$$



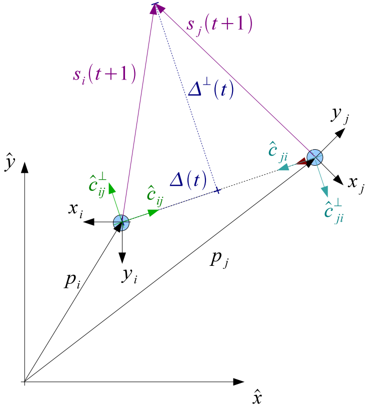
Interaction Rule $\mathcal{R}$ (VI)

If two agents (i, j) are selected at time t, their local estimate is updated as follows:

$$s_i(t+1) = \Delta(t) \cdot \hat{c}_{ij} + \Delta^{\perp}(t) \cdot \hat{c}_{ij}^{\perp},$$
$$s_j(t+1) = \Delta(t) \cdot \hat{c}_{ji} + \Delta^{\perp}(t) \cdot \hat{c}_{ji}^{\perp}.$$

where:

$$\Delta(t) = \frac{d_{ij} - s_j(t)^T \hat{c}_{ji} + s_i(t)^T \hat{c}_{ij}}{2},$$
$$\Delta^{\perp}(t) = \frac{s_i(t)^T \hat{c}_{ij}^{\perp} - s_j(t)^T \hat{c}_{ji}^{\perp}}{2},$$



Remarks (I)

A couple of remarks are now in order:

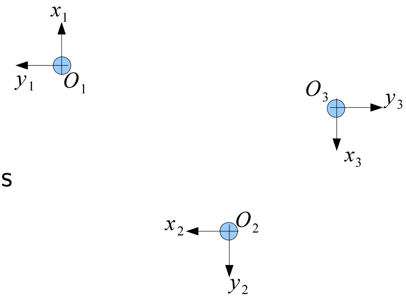
- This update rule leads itself to an easy decentralized implementation of the algorithm,
- All the parameters are local to the agents and independent to any specific reference frame.
- Agents estimate their relative position $d_{ij} = \|p_i - p_j\|$ and the line of sight $\hat{c}_{ij} = \frac{p_i - p_j}{\|p_i - p_j\|}$ both in their own local reference frame.

Agreement on a Common Reference Frame (I)

Our main objective is to have the multi-agent system reach an agreement on a common reference frame (CRF).

Steps

1. Agreement on a set of two common points $\mathcal{F}_i = \{f_{1,i}, f_{2,i}\}$.
2. Construction of a CRF $\mathcal{O}_r = \{r_x,\ r_y\}$.
3. Construction of a homogeneous transformation matrix A_i^r from $\mathcal{O}_i$ to $\mathcal{O}_r$.

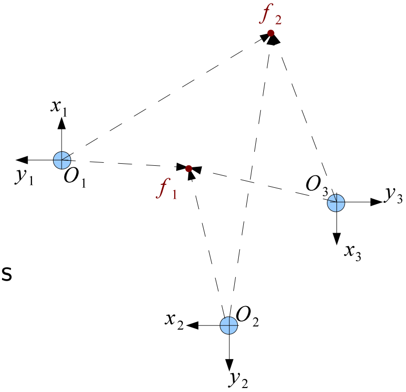


Agreement on a Common Reference Frame (II)

Our main objective is to have the multi-agent system reach an agreement on a common reference frame (CRF).

Steps

1. Agreement on a set of two common points $\mathcal{F}_i = \{f_{1,i}, f_{2,i}\}$.
2. Construction of a CRF $\mathcal{O}_r = \{r_x,\ r_y\}$.
3. Construction of a homogeneous transformation matrix A_i^r from $\mathcal{O}_i$ to $\mathcal{O}_r$.

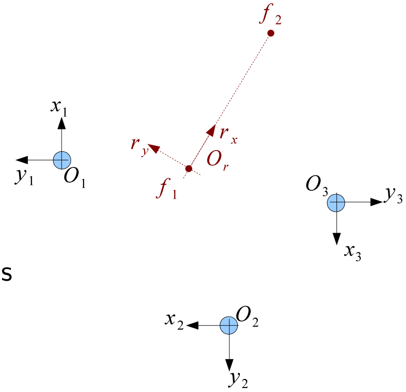


Agreement on a Common Reference Frame (III)

Our main objective is to have the multi-agent system reach an agreement on a common reference frame (CRF).

Steps

1. Agreement on a set of two common points $\mathcal{F}_i = \{f_{1,i}, f_{2,i}\}$.
2. Construction of a CRF $\mathcal{O}_r = \{r_x, r_y\}$.
3. Construction of a homogeneous transformation matrix A_i^r from $\mathcal{O}_i$ to $\mathcal{O}_r$.

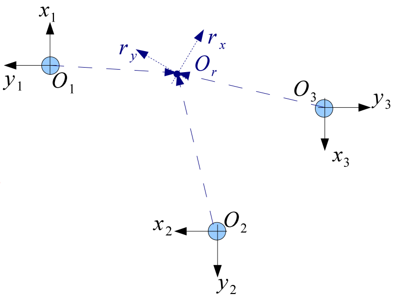


Agreement on a Common Reference Frame (IV)

Our main objective is to have the multi-agent system reach an agreement on a common reference frame (CRF).

Steps

1. Agreement on a set of two common points $\mathcal{F}_i = \{f_{1,i}, f_{2,i}\}$.
2. Construction of a CRF $\mathcal{O}_r = \{r_x,\ r_y\}$.
3. Construction of a homogeneous transformation matrix A_i^r from $\mathcal{O}_i$ to $\mathcal{O}_r$.



Agreement on a Common Reference Frame (II)

Algorithm 1: Reference Frame Agreement Algorithm

Data: $\mathcal{F}_i = \{f_{1,i}, f_{2,i}\}$
Result: R_i^r

- Compute the versors $r_{x,i}$ and $r_{y,i}$:

$$r_{x,i} = \frac{(f_{2,i} - f_{1,i})}{\|f_{2,i} - f_{1,i}\|} \qquad r_{y,i} = r_{x,i}^{\perp},$$

- Compute the translation vector t_i:

$$t_i = \|f_{1,i} - p_i\|,$$

- Compute the homogeneous transformation matrix A_i^r:

$$A_i^r = \begin{bmatrix} R_i^r & t_i \\ 0 & 1 \end{bmatrix}$$

Outline

Gossip Algorithm (I)

Definition (1)

Let us define $\mathbb{G}(t, t+\Delta t) = \{V, \mathbb{E}(t, t+\Delta t)\}$, where $\mathbb{E}(t, t+\Delta t) = \bigcup_{k=t}^{t+\Delta t} \mathfrak{e}(k)$, as the graph resulting from the union of all the edges given by the edge selection process from time t to $t+\Delta t$.

Definition (2 - $\mathcal{S}$)

Let $\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$, with $s_i \in \mathbb{R}^2, \forall i = 1, \ldots, n$ be the set of current agents local estimates, each one in their own reference frame.

Gossip Algorithm (II)

Definition (3 - $\mathcal{R}$)

- Let

$$\begin{aligned} \Delta(t) &= \frac{d_{ij} - s_j(t)^T \hat{c}_{ji} + s_i(t)^T \hat{c}_{ij}}{2}, \\ \Delta^{\perp}(t) &= \frac{s_i(t)^T \hat{c}_{ij}^{\perp} - s_j(t)^T \hat{c}_{ji}^{\perp}}{2}, \end{aligned} \quad (5)$$

- $\mathcal{R}$:

$$\begin{aligned} s_i(t+1) &= \Delta(t) \cdot \hat{c}_{ij} + \Delta^{\perp}(t) \cdot \hat{c}_{ij}^{\perp}, \\ s_j(t+1) &= \Delta(t) \cdot \hat{c}_{ji} + \Delta^{\perp}(t) \cdot \hat{c}_{ji}^{\perp}. \end{aligned} \quad (6)$$

Agreement on a Common Point

Theorem (1)

Let us consider a gossip algorithm $\{\mathcal{S}, \mathcal{R}, \mathfrak{e}\}$, with $\mathcal{S}, \mathcal{R}$ defined respectively as in Definition (2), and Definition (3). If $\mathfrak{e}$ is such that $\forall t,\ \exists\ \Delta t\ : \mathbb{G}(t, t+\Delta t)$ is connected, then:

$$\lim_{t\to\infty} s_{gi}(t) = R_i s_i(t) + p_i = \frac{1}{n}\sum_{i=1}^{n}\Big(R_i s_i(0) + p_i\Big), \quad \forall\, i = 1, \ldots, n. \tag{7}$$

Note: The agreement depends upon the initial set of local agents estimate $S_0 = \{s_1(0), s_2(0), \cdots, s_n(0)\}$.

Agreement on the Multi-Agent System Centroid

Corollary (1)

Let us consider the gossip algorithm defined by $\{\mathcal{S}, \mathcal{R}, \mathfrak{e}\}$ *as in Theorem 5. If each agent initializes its state* $s_i(0) = 0$ *to zero, then all the agents estimates converge to the network centroid:*

$$\lim_{t \to \infty} s_{gi}(t) = R_i s_i(t) + p_i = \frac{1}{n} \sum_{i=1}^{n} p_i, \quad \forall\, i = 1, \ldots, n. \qquad (8)$$

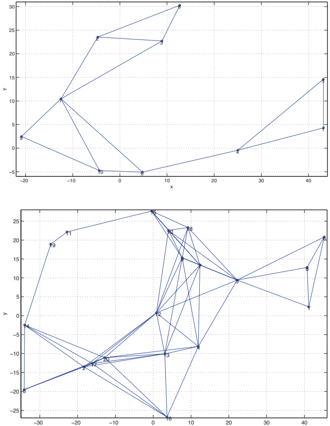
Outline

Simulation Setup

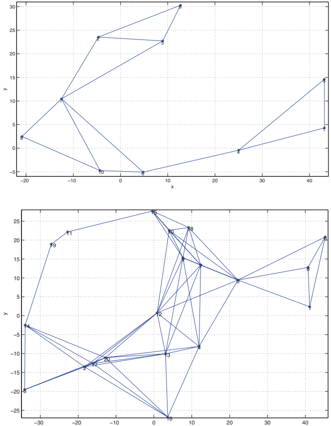
- Simulations have been carried out by exploiting a framework developed in Matlab.
- Only simulations concerning the agreement on a common point, i.e., multi-agent system centroid, are here reported.
- Two different scenarios have been considered:
  - Perfect Measurements.
  - Noisy Measurements.

Perfect Measurements vs. Noisy Measurements

Perfect Measurements



Noisy Measurements



Algorithm Robustness to Noise

Experimental results have shown that:

1. The proposed method is inherently robust against noise in the distance measurements:
   - The effect of the noise can be locally averaged.
   - The effect of the noise results in a symmetric contribution.
   - The local estimation is perturbed but the final converging point is not affected.
2. The proposed method is not robust against noise in the measurements with respect to the direction of the line of sight:
   - The effect of the noise results in a non symmetric contribution.
   - The propagation of the noise is not linear.
   - The inaccuracy may indeed move the convergence point.

Outline

Results Wrap-Up & Future Work

What we have done so far:

- The problem of decentralized agreement has been addressed
- An algorithm to perform an agreement toward a common point has been proposed.
- A theoretical analysis of the convergence properties has been provided.
- An experimental validation has been carried out.

What we still have to do:

- A theoretical validation of the empirical evidences concerning noisy measurements.
- A theoretical analysis of the converge properties modeling disturbances.

Any questions?

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Theoretical Analysis (I)

Lemma (1)

The proposed gossip algorithm $\{\mathcal{S}, \mathcal{R}, \mathfrak{e}\}$ *can be equivalently stated with respect to a global common reference frame as follows:*

$$\begin{aligned} x(t+1) &= W(\mathfrak{e}(t))x(t), \\ y(t+1) &= W(\mathfrak{e}(t))y(t), \end{aligned} \tag{9}$$

where $s_{gi}(t) = [x_i(t)\ y_i(t)]^T$ *with* $x(t) = [x_1(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ *and* $y(t) = [y_1(t), \ldots, y_n(t)]^T \in \mathbb{R}^n$, *and* $W(\mathfrak{e}(t))$ *is a matrix representation of the update rule* $\mathcal{R}$ *defined as:*

$$W(e_{ij}) = I - \frac{(\hat{e}_i - \hat{e}_j)(\hat{e}_i - \hat{e}_j)^T}{2}. \tag{10}$$

Theoretical Analysis (II)

Lemma (2)

If $\mathfrak{e}$ is such that $\forall t,\ \exists\ \Delta t\ :\mathbb{G}(t, t+\Delta t)$ *is connected, then:*

$$\hat{C}_{(t,t+\Delta t)} = \bigcap_{e_{ij}\in\mathbb{E}(t,t+\Delta t)} C(e_{ij}) = span\{\mathbf{1}_n\}, \qquad (11)$$

where $\mathbf{1}_n = [1, \dots, 1]^T$ *is a* $n \times 1$ *unit vector with all components equal to 1, and* $C(e_{ij})$ *is the set of fixed points related to* $W(e_{ij})$ *defined as:*

$$C(e_{ij}) = Fix\ W(e_{ij}) = \{x \in \mathbb{R}^n : W(e_{ij})\,x = x\}.$$

Theoretical Analysis (III)

Lemma (3)

If $\mathfrak{e}$ is such that $\forall t,\ \exists\ \Delta t\ : \mathbb{G}(t, t+\Delta t)$ is connected, then there exists a norm such that:

$$\|W(e_{ij})\,x - c\| \leq \|x - c\|, \qquad \forall\, c \in \hat{C}_{(t,t+\Delta t)}, \quad \forall\, e_{ij} \in \mathbb{E}_{(t,t+\Delta t)},\ \forall\, x \in \mathbb{R}^n \tag{12}$$

$$\|\Phi_{(t,t+\Delta t)}\,x - c\| < \|x - c\|, \qquad \forall\, c \in \hat{C}_{(t,t+\Delta t)}, \quad \forall\, x \in \mathbb{R}^n \setminus \hat{C}_{(t,t+\Delta t)} \tag{13}$$

where $\Phi_{(t,t+\Delta t)} = \prod_{e_{ij} \in \mathbb{E}(t,t+\Delta t)} W(e_{ij})$.

Theoretical Analysis (IV)

Lemma (4)

If $\mathfrak{e}$ *is such that* $\forall t, \; \exists \; \Delta t \; : \mathbb{G}(t, t+\Delta t)$ *is connected, then for any sequence of intervals* $\{I_i\}$ *where* $I_i = I_{i-1} + \Delta t_i$ *with* $I_0 = 0$ *and* $I_j > I_i \; \forall \, j > i$*, it holds:*

$$d\left(x(I_i), span\{\mathbf{1}_n\}\right) \rightarrow 0. \tag{14}$$

Theoretical Analysis (V)

Theorem (1)

Let us consider a gossip algorithm $\{\mathcal{S}, \mathcal{R}, \mathfrak{e}\}$*, with* $\mathcal{S}, \mathcal{R}$ *defined respectively as in Definition (2), and Definition (3). If* $\mathfrak{e}$ *is such that* $\forall t,\ \exists\ \Delta t\ : \mathbb{G}(t, t + \Delta t)$ *is connected, then:*

$$\lim_{t \to \infty} s_{gi}(t) = R_i s_i(t) + p_i = \frac{1}{n} \sum_{i=1}^{n} \Big(R_i s_i(0) + p_i \Big), \qquad \forall\, i = 1, \ldots, n. \tag{15}$$

Theoretical Analysis (VI)

Proof Sketch (I).

The theorem can be proven by exploiting the Lemmas previously introduced. In particular,

1. By using Lemma 1 the gossip algorithm can be investigated independently for each axis:

$$
\begin{aligned}
x(t+1) &= W(\mathfrak{e}(t))x(t), \\
y(t+1) &= W(\mathfrak{e}(t))y(t),
\end{aligned}
$$

2. By using Lemma 2 we know that for any given interval $[t, t+\Delta t]$:

$$
\hat{C}_{(t,t+\Delta t)} = \bigcap_{e_{ij} \in \mathbb{E}(t,t+\Delta t)} C(e_{ij}) = \text{span}\{\mathbf{1}_n\}.
$$

□

Theoretical Analysis (VII)

Proof Sketch (II).

3. By using Lemma 3, we know that for any given interval $[t, t+\Delta t]$ such that $\mathcal{G}(t+\Delta t)$ is connected the following holds:

$$\|\Phi_{(t,t+\Delta t)}\, x - c\| < \|x - c\|, \forall\, c \in \hat{C}_{(t,t+\Delta t)},\ \forall\, x \in \mathbb{R}^n \backslash \hat{C}_{(t,t+\Delta t)}$$

4. By using Lemma 4, we know that exists a sequence of intervals $\{I_i\}$ so that:

$$d\left(x(I_i), \mathrm{span}\{\mathbf{1}_n\}\right) \rightarrow 0.$$

Therefore, the sequence $\{x(I_i)\}$ converges in norm to some points in $\mathrm{span}\{\mathbf{1}_n\}$, that is

$$\|x(I_i) - c\| \rightarrow 0 \qquad \text{then} \qquad \{x(I_i)\} \rightarrow c, \quad c \in \mathrm{span}\{\mathbf{1}_n\}.$$

□

Theoretical Analysis (VIII)

Proof Sketch (III).

In addition, each single matrix $W(e_{ij})$ is a symmetric row-sum matrix:

$$\mathbf{1}_n^T W(e_{ij}) = \mathbf{1}_n^T \qquad W(e_{ij})\mathbf{1}_n = \mathbf{1}_n.$$

Therefore, the sum of the vector components must be preserved over time at each iteration. This implies that for a given $c = \gamma\,\mathbf{1}_n$:

$$\sum_{i=1}^{n} c_i = \sum_{i=1}^{n} x_i(l_0), \quad \gamma = \frac{\sum_{i=1}^{n} x_i(l_0)}{n}.$$

□

Theoretical Analysis (IX)

Proof Sketch (IV).

From this it follows that:

$$x(l_i) \to \frac{\sum_{i=1}^{n} x_i(l_0)}{n} \mathbf{1}_n, \quad \text{thus} \quad y(l_i) \to \frac{\sum_{i=1}^{n} y_i(l_0)}{n} \mathbf{1}_n.$$

Therefore, for each agent i we have:

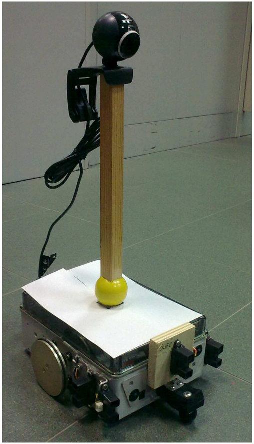
$$s_{gi}(t) \to \left[\frac{\frac{\sum_{i=1}^{n} x_i(l_0)}{n}}{\frac{\sum_{i=1}^{n} y_i(l_0)}{n}} \right] = \frac{1}{n} \sum_{i=1}^{n} \Big(R_i s_i(0) + p_i \Big),$$

□

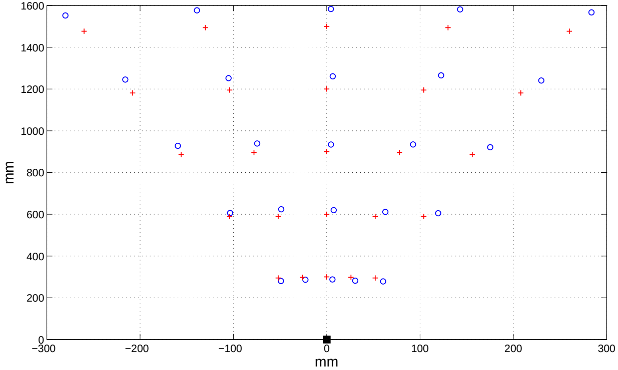
Experimental Setup (WIP)

For the experiments the following assumptions are made:

- Each robot is equipped with a webcam
- Each robot is uniquely identifiable by means of a color ID



Webcam Calibration



- Range error grows as the distance increases.
- Angular error is sufficiently small to be neglected.

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