Evaluating Network Rigidity in Realistic Systems: Decentralization, Asynchronicity, and Parallelization

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Abstract—In this paper, we consider the problem of evaluating the rigidity of a planar network, while satisfying common objectives of real-world systems: decentralization, asynchronicity, and parallelization. The implications that rigidity has in fundamental multi-robot problems, e.g., guaranteed formation stability and relative localizability, motivates this work. We propose the decentralization of the pebble game algorithm of Jacobs et. al., an $O(n^2)$ method that determines the generic rigidity of a planar network. Our decentralization is based on asynchronous messaging and distributed memory, coupled with auctions for electing leaders to arbitrate rigidity evaluation. Further, we provide a parallelization that takes inspiration from gossip algorithms to yield significantly reduced execution time and messaging. An analysis of the correctness, finite termination, and complexity is given, along with a simulated application in decentralized rigidity control. Finally, we provide Monte Carlo analysis in a Contiki networking environment, illustrating the real-world applicability of our methods, and yielding a bridge between rigidity theory and realistic interacting systems.

Index Terms—Networked Robots; Distributed Robot Systems; Asynchronous and Parallel Communication; Graph Rigidity.

I. INTRODUCTION

MULTI-ROBOT networks remain among the areas of interest at the forefront of robotics research, particularly given the steady advancement of wireless communication, embedded computation, and hardware platforms. Intuitively, networks of intelligently interacting robots provide significant advantages over the single-agent alternative; for example scalability, failure robustness, spatiotemporal efficiency, heterogeneity, etc. As recent work has demonstrated, multi-robot investigations are far-reaching across various disciplines, ranging from sampling, tracking, and coverage [1]–[3], mobility and topology control [4]–[6], to general agent agreement problems [7]–[9].

In modeling and analyzing multi-robot networks, research balances between accuracy in approximating realistic systems, and ease of technical analysis, most typically in understanding mobility, communication, and sensing. Here we take the former approach, considering a problem that underlies fundamental objectives in multi-robot research, while operating under the commonly desired parameters of real-world systems: decentralized implementation where information is exchanged only in local neighborhoods, asynchronicity in communication, and parallelization of agent actions to maximize efficiency. Our problem of interest in this work is the evaluation of the rigidity property of an interconnected system of intelligent agents, e.g., robots, sensors, etc. A relatively under-explored topic in the area of multi-agent systems, rigidity has important implications particularly for mission objectives requiring collaboration. For example, rigidity is vital for guaranteeing stability in controlling formations of mobile vehicles, when only relative inter-agent information is available [10]–[15]. Further, when a global frame of reference is inaccessible, rigidity becomes a necessary and under certain conditions sufficient condition for localization tasks with distance or bearing-only measurements [16]–[19]. Rigidity is also a necessary component of global rigidity [20]–[22], which can further strengthen the guarantees of formation stability and localizability. We point out that it is typical in the literature to assume rigidity properties of the network in order to achieve multi-agent behaviors, however few works provide means of evaluating or achieving network rigidity in a dynamic manner, or under the network conditions considered here.

The general study of rigidity has a rich history in various contexts of science, mathematics, and engineering [21]–[28]. In [27], combinatorial operations are defined which preserve rigidity, with works such as [10], [12] extending the ideas to multi-robot formations. In [29] an algorithm is proposed for generating rigid graphs in the plane based on the Henneberg construction [27], however from a centralized perspective. Similarly, [30] defines decentralized rigid constructions that are edge length optimal, however provide no means of determining an unknown graph’s rigidity properties. The work [31] defines a rigidity eigenvalue for infinitesimal rigidity evaluation and control, however such efforts remain centralized and require continuous communication and computational resources.

As opposed to previous work, we propose a decentralized method of evaluating generic graph rigidity in the plane, without a priori topological information, to our knowledge the first such effort, particularly in a multi-agent context. To this end, we decentralize in an asynchronous manner the pebble game proposed by Jacobs and Hendrickson in [28], an algorithm that determines in $O(n^2)$ time the combinatorial rigidity of a network, and a spanning edge set defining the minimally rigid subcomponent of the graph. Specifically, we propose a leader election procedure based on distributed auctions that manages the sequential nature of the pebble game in
a decentralized setting, together with a distributed memory architecture. Further, an asynchronous messaging scheme preserves local-only agent interaction, as well as robustness to delays, failures, etc. Towards network efficiency, we extend our decentralization by parallelizing a portion of the rigidity evaluation, taking inspiration from gossip messaging, yielding significant improvements in execution time and communication. To illustrate our contributions, we provide a thorough analysis of the correctness, finite termination, and complexity of our propositions, along with an illustration of decentralized rigidity control. Finally, we provide Monte Carlo analysis of our algorithms in a Contiki networking environment, illustrating the real-world applicability of our methods.

Although a few recent works have begun to investigate rigidity evaluation or control \[29\]–\[32\], they provide graph constructions or centralized control relying on expensive estimation techniques. We seek decentralization specifically to enhance scalability and robustness as network size increases, and to serve systems where centralized operation may be difficult or impossible. Our contributions therefore aim to bridge the gap between fundamentally important multi-agent behaviors and realistic rigidity evaluation in networked systems, ultimately moving towards robotic/sensor systems with achievable rigidity-based behaviors (as in our previous work \[33\]).

In summary, the major contributions of this paper are as follows:

- A leader-based, asynchronous decentralization of the classically centralized and serial pebble game algorithm, yielding a decentralized method for planar rigidity evaluation.
- A study of exploiting structural properties of rigidity for parallelization of our decentralized algorithm.
- A characterization of our algorithms in the real-world Contiki networking environment, with a full codebase release for use in the robotics community.

A preliminary portion of this paper appeared in \[34\], compared to which we provide expanded analysis and correctness proofs, a complete treatment of parallelization, expanded rigidity control simulations, and a Monte Carlo analysis in the Contiki environment demonstrating applicability under realistic conditions.

The outline of the paper is as follows. In Section II we provide preliminary materials including agent and network models, a model of algorithm execution, and primers on rigidity theory and the pebble game. A decentralization of the pebble game is presented in Section III with a parallelization given in Section IV. Simulation results are provided in Section V, with concluding remarks as well as directions for future work are stated in Section VI. Finally, technical algorithm details and related proofs are given in the Appendix.

## II. PRELIMINARIES AND FORMULATION

### A. Agent, Network, and Execution Models

Consider a system composed of \( n \) agents indexed by \( \mathcal{I} = \{1, \ldots, n\} \) operating in \( \mathbb{R}^2 \), each possessing computation and communication capabilities, denoting by \( (i, j) \) a bi-directional communication link between agents \( i \) and \( j \). The agent may also be mobile, and thus we assume basic continuous dynamics

\[
\dot{x}_i = f(x) \tag{1}
\]

where \( x_i, f(x_i) \in \mathbb{R}^2 \) are the position and the velocity control input for an agent \( i \in \mathcal{I} \), respectively, and \( x \in \mathbb{R}^{2n} \) is the vector of stacked agent positions. For the purposes of integration into a motion control architecture (Section V-A), it is further assumed that each robot can sense other nearby robots and obstacles, yielding the displacement \( d_{ij} \in \mathbb{R}^2 \equiv \|x_{ij}\| \equiv \|x_i - x_j\| \).

To describe the interconnected system formally, we define an undirected graph \( G = (V, E) \), having vertices \( V = \{v_1, \ldots, v_n\} \) associated with each agent \( i \in \mathcal{I} \), and edge set \( E = \{(i, j) \mid i, j \in V\} \) with unordered pairs \( (i, j) \), where by definition \( (i, j) \in E \iff (j, i) \in E, \forall i \neq j \in \mathcal{I} \), excluding the possibility for self loops, \( (i, i) \notin E, \forall i \in \mathcal{I} \). Agents \( i \) and \( j \) with an edge \( (i, j) \in E \) are referred to as neighbors, where the set of neighbors for the \( i \)-th agent is given by \( \mathcal{N}_i = \{v_j \in V \mid (i, j) \in E\} \).

**Assumption 1 (Connectedness):** We assume the network topology \( G \) is connected for all time to guarantee all agents can participate in rigidity evaluation (Sections III and IV), that is for every pair of nodes \( i, j \) there exists a sequence of nodes in \( G \) that are adjacent and connect \( i, j \). Notice that this assumption is trivially satisfied in rigid networks.

As our concern in this work is to operate under the typical parameters of realistic interacting systems, we assume an asynchronous model of time, where each agent \( i \in \mathcal{I} \) has a clock which ticks according to some discrete distribution with finite support \( \mathbb{N} \) independently of the clocks of the other agents \( \mathbb{N} \), allowing also for the possibility of delayed communication over links \( (i, j) \in E \). Equivalently, this corresponds to a global clock having time-slots \( [t_k, t_{k+1}) \) which discretizes system time according to clock ticks, where for convenience we will use simply \( t \) to denote time \( \mathbb{N} \). Such assumptions induce asynchronicity in both agent computation and the broadcast and reception of inter-agent messages. First, we make the following assumptions concerning agent execution:

**Assumption 2 (Agent execution):** Each agent \( i \in \mathcal{I} \) executes according to an algorithm on ticks of their clock, handling messages from neighbors \( j \in \mathcal{N}_i \) and sending messages if dictated by the execution. Local execution is assumed to consist of atomic logic and message handling, that is all local algorithmic state, denoted \( X_i \), is assumed to be without race conditions due to asynchronicity.

A coordinated algorithm execution with associated stopping condition can then be defined as follows:

**Definition 2.1 (Coordinated execution):** A coordinated algorithm execution is given as a sequence of ticks \( t_k \) and therefore local execution and asynchronous messaging, yielding a terminal state upon some discrete network stopping condition

\[
f_{\text{stop}} \in \{0, 1\} \equiv f((X_1, \ldots, X_n)) \tag{2}
\]

dependent on the execution states of the network agents. It is assumed that \( f \) can be computed using distributed techniques, e.g., consensus \[37\], as will be demonstrated in our proposed...
algorithms. After the stopping condition is observed the agents enter into an idle state where no execution occurs.

Finally, to guarantee soundness with respect to network communication and asynchronicity, we make the following assumptions:

Assumption 3 (Asynchronous messaging): We assume each agent \( i \in \mathcal{I} \) treats messages received from neighbors \( j \in \mathcal{N}_i \) in a first-in-first-out (FIFO) manner, guaranteeing soundness with respect to our proposed algorithm executions. Further, the possibility of communication failure is handled with best-effort messaging, i.e., there exists an underlying communication control layer where a best effort is made to deliver packets in the network. Specifically, we assume that the best effort guarantees message reception in finite time, or equivalently a message failure can be handled appropriately with respect to the algorithms that will be discussed.

B. Rigidity Theory

The primary concern of this work is the rigidity property of the underlying graph \( G \) describing the network topology, specifically as rigid graphs imply guarantees for example in both localizability and formation stability of multi-robot systems [23]. To begin, we recall the intuition of how rigidity is recognized in a planar graph, following the exposition of [28]. Clearly, graphs with many edges are more likely to be rigid than those with only a few, specifically as each edge acts to constrain the degrees of freedom of motion of the agents in the graph. In \( \mathbb{R}^2 \), there are \( 2n \) degrees of freedom in a network of \( n \) agents, and when we remove the three degrees associated with rigid translation and rotation, we arrive at \( 2n - 3 \) degrees of freedom we must constrain to achieve rigidity. Each edge in the graph can be seen as constraining these degrees of freedom, and thus we expect \( 2n - 3 \) edges will be required to guarantee a rigid graph. In particular, if a subgraph containing \( k \) vertices happens to contain more than \( 2k - 3 \) edges, then these edges cannot all be required for constraining the degrees of motion, i.e., they cannot be all independent. Our goal in evaluating rigidity is thus to identify the \( 2n - 3 \) edges that independently constrain the motion of our agents, precisely describing a networks underlying rigid component.

We now provide a brief technical overview of the above intuition, and direct the reader to [23]–[27] for an in depth review of rigidity theory. First, we require the notion of a graph embedding in the plane, captured by the framework \( \mathbb{F}_p \equiv (G, p) \) comprising graph \( G \) together with a mapping \( p: V \rightarrow \mathbb{R}^2 \), assigning to each node in \( G \), a location in \( \mathbb{R}^2 \). The natural embedding for us is to assign each node the position \( x_i \), associated with each agent, defined by the mapping \( p(i) = x_i \), otherwise known as a realization of \( G \) in \( \mathbb{R}^2 \). Therefore, a framework describes both the communication topology of a multi-agent system, and the spatial configuration of each agent in the plane.

The infinitesimal motion of \( \mathbb{F}_p \) can be described by assigning to the vertices of \( G \), a velocity \( \dot{x}_i \in \mathbb{R}^2 \), such that

\[
(\dot{x}_i - \dot{x}_j) \cdot (x_i - x_j) = 0, \quad \forall (i, j) \in E
\]

where \( \cdot \) is the standard dot product over \( \mathbb{R}^m \). That is, edge lengths are preserved, implying that no edge is compressed or stretched over time. The framework is said to undergo a finite flexing if \( p_i \) is differentiable and edge lengths are preserved, with trivial flexings defined as translations and rotations of \( \mathbb{R}^2 \) itself. If for \( \mathbb{F}_p \) all infinitesimal motions are trivial flexings, then \( \mathbb{F}_p \) is said to be infinitesimally rigid. Otherwise, the framework is called infinitesimally flexible, as in Fig. 1a where \( v_1 \) and \( v_3 \) can move inward with \( v_2 \) and \( v_4 \) moving outward, while preserving edge lengths [27]. In the context of a robotic network, rigid infinitesimal motion corresponds to movement of the ensemble in which the distances between robots remain fixed over time.

The infinitesimal rigidity of \( \mathbb{F}_p \) is tied to the specific embedding of \( G \) in \( \mathbb{R}^2 \), however it has been shown that the notion of rigidity is a generic property of \( G \), specifically as almost all realizations of a graph are either infinitesimally rigid or flexible, i.e., they form a dense open set in \( \mathbb{R}^2 \) [38]. Thus, we can treat rigidity from the perspective of \( G \), abstracting away the necessity to check every possible realization. The first such combinatorial characterization of graph rigidity was described by Laman in [23], and is summarized as follows:

**Theorem 2.1 (Graph rigidity, [23]):** A graph \( G = (\mathcal{V}, \mathcal{E}) \) with realizations in \( \mathbb{R}^2 \) having \( n \geq 2 \) nodes is rigid if and only if there exists a subset \( \mathcal{E} \subseteq \mathcal{E} \) consisting of \( |\mathcal{E}| = 2n - 3 \) edges satisfying the property that for any non-empty subset \( \hat{\mathcal{E}} \subseteq \mathcal{E} \), we have \( |\hat{\mathcal{E}}| \leq 2k - 3 \), where \( k \) is the number of nodes in \( \mathcal{V} \) that are endpoints of \( (i, j) \in \mathcal{E} \).

Laman’s notion of graph rigidity is also referred to as generic rigidity, and is characterized by the Laman conditions on the network’s subgraphs. Intuitively, the concept of rigidity can be thought of in a physical way, that is if the graph were a bar and joint framework, it would be mechanically rigid against...
external and internal forces. However, we point out that the rigidity of an underlying graph is purely a topological property. A network of agents that is described by a rigid graph is not necessarily mechanically rigid. Instead, the structure of its interconnections, in our case robot-to-robot communication, possesses the combinatorial properties of the above Laman conditions.

Denote by $G_{\mathbb{R}}$ the set of all rigid graphs in $\mathbb{R}^2$, and the graph $\mathcal{S} = (\mathcal{V}, \mathcal{E})$ satisfying Theorem 2.1 a Laman subgraph of $G$. It follows from Theorem 2.1 that any rigid graph in the plane must then have $|\mathcal{E}| \geq 2n - 3$ edges, with equality for minimally rigid graphs. The impact of each edge on the rigidity of $G$ is captured in the notion of edge independence, a direct consequence of Theorem 2.1.

Definition 2.2 (Edge independence, [28]): Edges $(i, j) \in \mathcal{E}$ of a graph $G = (\mathcal{V}, \mathcal{E})$ are independent in $\mathbb{R}^2$ if and only if no subgraph $\bar{G} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$ has $|\bar{\mathcal{E}}| > 2|\bar{\mathcal{V}}| - 3$. A set of independent edges will be denoted by $\mathcal{E}^*$, while the graph over $\mathcal{E}^*$ is denoted by $G^*$.

The above conditions imply that a graph is rigid in $\mathbb{R}^2$ if and only if it possesses $|\mathcal{E}^*| = 2n - 3$ independent edges, where edges that do not meet the conditions of Definition 2.2 are called redundant. Thus, in determining the rigidity of $G$, we must verify the Laman conditions to discover a suitable set of independent edges $\mathcal{E}^*$. We refer the reader Fig. 1 for a depiction of graph rigidity. Notice that the graph in Fig. 1a is non-rigid as it does not fulfill the basic condition of Definition 2.2.

To lessen the exponential complexity of the Laman conditions, we consider the pebble game proposed by Jacobs and Hendrickson in [28]. A brief overview of the centralized pebble game will be given here, beginning with a useful characterization of the Laman conditions and edge independence:

Theorem 2.2 (Laman restated, [28]): For graph $G = (\mathcal{V}, \mathcal{E})$, the following statements are equivalent:

- All $(i, j) \in \mathcal{E}$ are independent in $\mathbb{R}^2$.
- For each $(i, j) \in \mathcal{E}$, the graph formed by quadrupling $(i, j)$, i.e., adding 4 virtual copies of $(i, j)$ to $\mathcal{E}$, has no subgraph $\bar{G} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$ in which $|\bar{\mathcal{E}}| > 2|\bar{\mathcal{V}}|$.

Lemma 2.1 (Edge quadrupling, [28]): Given an independent edge set $\mathcal{E}^*$, an edge $(i, j) \notin \mathcal{E}^*$ is independent of $\mathcal{E}^*$ if and only if the graph formed by the union of $\mathcal{E}^*$ and quadrupled edge $(i, j)$ has no subgraph $\bar{G} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$ in which $|ar{\mathcal{E}}| > 2|\bar{\mathcal{V}}|$.

The above Lemma provides us with a simple process for testing rigidity: we incrementally quadruple edges in the graph, check the induced subgraph property, and continue until we have either discovered $2n - 3$ independent edges or we have exhausted $\mathcal{E}$. However, this process alone does not save us from the exponential complexity of verifying the subgraph property. To this end, [28] provides a natural simplification in the following pebble game:

Definition 2.3 (The pebble game, [28]): Considering a graph $G = (\mathcal{V}, \mathcal{E})$ where we associate an agent with each $v \in \mathcal{V}$, give to each agent two pebbles which can be assigned to an edge in $\mathcal{E}$. Our goal in the pebble game is to assign the pebbles in $G$ such that all edges are covered, i.e., a pebble covering. In finding a pebble covering, we allow the assignment of pebbles by agent $i$ only to edges incident to $v_i$ in $G$. Further we allow pebbles to be rearranged only by removing pebbles from edges which have an adjacent vertex with a free pebble, such that the free pebble is shifted to the assigned pebble, freeing the assigned pebble for assignment elsewhere. Thus, if we consider pebble assignments as directed edges exiting from an assigning agent $i$, a pebble search over a directed network occurs. If a free pebble is found, the rules for local assignment and rearranging then dictate the pebble’s return and assignment to $(i, j)$.

Lemma 2.2 (Pebble covering, [28]): In the context of the pebble game of Definition 2.3 if there exists a pebble covering for an independent edge set $\mathcal{E}^*$ with a quadrupled edge $(i, j) \notin \mathcal{E}^*$, there is no subgraph violating the conditions of Lemma 2.1 and the set $\mathcal{E}^* \cup (i, j)$ is independent.

Rigidity evaluation now operates as follows: every edge $e \in \mathcal{E}$ is quadrupled, and an attempt to expand the current pebble covering for $\mathcal{E}^*$ to each copy of $e$ is made, with success resulting in $\mathcal{E}^* \leftarrow \mathcal{E}^* \cup e$ and termination coming when $|\mathcal{E}^*| = 2n - 3$. Intuitively, an agent’s pebbles represent

![Fig. 2. An example of the pebble game for a rigid graph with $n = 3$ with progression from left to right. Pebbles are given by black dots, quadrupled edges by thick links (blue), pebble shifts by dashed arrows, and the local assignment of pebbles by black arrows. Graph edges that remain to be quadrupled are dashed. We have here $|E^*| = 3$.](image-url)
its possible commitments to the network’s subgraphs, while maintaining the subgraph conditions of Lemma 2.1 or in a physical way the degrees of freedom of motion in $\mathbb{R}^{k}$. Further, the edge quadrupling operation and the pebble game effectively cast the Laman conditions on subgraphs in terms of a matching problem. That is, as each agent is given 2 pebbles, and each of 4 instances of a considered edge must be assigned a pebble, we implicitly verify the $2k - 3$ edge condition of Laman when these 4 pebbles are found in a subgraph containing $k$ vertices. This is the case precisely because each previously considered edge is assigned a single pebble.

The centralized pebble game of Jacobs is depicted in Algorithm 1 with an illustration of the quadrupling and pebble search procedure depicted in Fig. 2. In the simple three node graph shown, there are six available pebbles that can only be assigned locally. Thus, in quadrupling each edge and finding four pebbles, the subgraph conditions of Laman are incrementally verified. The progression is given from left to right and clockwise in the figure, with pebbles given by black dots, quadrupled edges by thick links (blue), pebble shifts by dashed arrows, and the local assignment of pebbles by black arrows. Graph edges that remain to be quadrupled are dashed. Notice that in discovering the pebble to cover the final copy of a quadrupled incident edge, leadership then transfers to the next auction winner when the current leader’s neighborhood has been exhausted.

### Algorithm 1: The centralized pebble game [28]

1: procedure PebbleGame($G = (V, E)$)
2: Assign each $v_i$ two pebbles, $\forall i \in \mathcal{I}$
3: $E^* \leftarrow \emptyset$
4: for all $(i, j) \in E$ do
5: Quadruple $(i, j)$ over $G$
6: Search for 4 pebbles, originating from $v_i$ and $v_j$
7: if found then
8: Rearrange pebbles to cover quadrupled $(i, j)$
9: $E^* \leftarrow E^* \cup (i, j)$
10: if $|E^*| = 2|V| - 3$ then
11: return $E^*$
12: end if
13: end if
14: end for
15: end procedure

The primary considerations in decentralizing the pebble game of [28] lie in the sequential building of the independent edge set $E^*$, the storage of $E^*$ and associated pebble assignments over a distributed network, and the search and rearranging of pebbles throughout the network. To deal with these issues, we summarize the high level components of our decentralized:

- **Leader Election**: to control the sequential building of $E^*$, lead agents are elected through auctions to examine their incident edges for independence. In determining edge independence, pebbles are queried from the network through inter-agent messaging in order to cover each copy of a quadrupled incident edge. Leadership then transfers to the next auction winner when the current leader’s neighborhood has been exhausted.
- **Distributed Storage**: independent edges and pebble assignments are localized to each agent, effectively distributing network storage. We then rely on messaging and proper agent logic to support pebble searches and shifts.
- **Local Messaging**: as opposed to searching a centralized graph object for pebbles to establish edge independence, we endow the network with a pebble request/response messaging protocol to facilitate pebble searches.

Intuitively, our leader-based decentralization is an incremental rooting of pebble searches at appropriately elected network leaders, effectively partitioning rigidity evaluation as in Fig. 3. For convenience we will denote by $\mathcal{S}$ our decentralization of the serial pebble game of Algorithm 1.

In describing our algorithm we associate with each agent $i \in \mathcal{I}$ the following variables, with initialization indicated by $\leftarrow$:

$$P_i \leftarrow \emptyset$$: Pebble assignment set containing at most two edges $\{(i, j) \in E | j \in \mathcal{N}_i\}$, that is incident edges $(i, j)$ to which a pebble is associated. For convenience, we let $p_i = 2 - |P_i| \in \{0, 1, 2\}$ denote agent $i$’s free pebble count.

$$E_i^* \leftarrow \emptyset$$: Local independent edge set, containing edges $\{(i, j) \in E | j \in \mathcal{N}_i\}$ for which quadrupling and pebble covering succeeds. By construction $E^* = \bigcup_i E_i^*$.

#### A. Leader Election

An execution of the $\mathcal{S}$ algorithm begins when an agent detects network conditions that require rigidity evaluation, e.g., verifying link deletion to preserve rigidity. The initiating agent begins by triggering an auction for electing an agent in the network to become the leader. Specifically, to each agent $i \in \mathcal{I}$ we associate a bid for leadership $r_i = [i, b_i]$ with $b_i \in \mathbb{R}_{\geq 0}$ indicating the agent’s fitness in becoming the new leader, with $b_i = 0$ if agent $i$ has previously been elected as a leader, and $b_i \in \mathbb{R}_+$ otherwise. Denoting the local bid set by $\mathcal{R}_i = \{r_j | j \in \mathcal{N}_i \cup \{i\}\}$, the auction then operates according the following agreement process:

$$r_i(t^+) = \arg\max_{r_j \in \mathcal{R}_i} b_j$$  \hspace{1cm} (4)

where the notation $t^+$ indicates a transition in $r_i$ after all neighboring bids have been collected through messaging. As $G$ is assumed connected for all time, (4) converges uniformly to the largest leadership bid

$$r_i = \arg\max_{r_j \in (0)} b_j, \quad \forall i, j \in \mathcal{I}$$  \hspace{1cm} (5)

after some finite time [39], [40]. After convergence of (4) the winning agent then takes on the leadership role, with the previous leader relinquishing its status. The proposed auction mechanism allows us to decentralize the pebble game by assigning to each leader the responsibility of expanding $E_i^*$ by evaluating only their incident edges for independence. Also, notice that previous leaders are never reelected due to $b_i = 0$
would possess those edges that both establish rigidity and are current size of the independent edge set and pebble covering. Also, note that each leader receives the once that network edges are considered only incident edges. In initializing \(E\) the expansion of \(E\) beyond simply discovering the network’s rigidity property.

As the agent with the largest bid is elected, the bids dictate the order of election and expansion of the independent set sequentially that the \(order\) of election is meaningful. We characterize that relationship in the following:

**Proposition 3.1 (Initial leader edges):** All incident edges \(\{(i, j) \in E \mid j \in N_i\}\) belonging to an initial leader \(i\) are members of the independent set \(\{i, j\} \in E^*\).

**Proof:** For each edge, a new node \(v_j\) must be considered as no two edges of \(i\) can have the same endpoint and \(E^*\) is empty due to \(i\) being the initial leader. Therefore, every subgraph containing the edges and nodes incident to \(i\) must have \(|E_s| \leq 2|V_s|-3\) edges, where \(V_s\) are the nodes of the considered subgraph and \(|E_s| = |V_s|-1\) due to the subgraph’s implicit tree structure. Thus, as there exists no subgraph violating Definition \(2.2\) the result follows.

As the agent with the largest bid is elected, the bids dictate the \(order\) of elected leaders and thus the edges that constitute the identified rigid subgraph. In other words, the bids can be applied based on the application. For example, if we assume each edge is assigned a weight which indicates its value in sensing or information, we could choose leader bids that are the sum of incident edge value. Then the resulting rigid subgraph would possess those edges that both establish rigidity and are the most valuable in the given application. Bids could also be chosen to reflect agent availability, processing capability, or the cardinality of incident edges, or they can be leveraged in terms of metrics related to mission objectives. The proposed auction technique therefore affords us control over \(E^*\) that goes beyond simply discovering the network’s rigidity property.

**B. Leader Tasks**

After election, the primary task of the leader \(i\) is to continue the expansion of \(E^*\) by evaluating the independence of each edge \((i, j) \in E_i \triangleq \{N_i \mid \neg \text{beenLeader}(j)\}\), i.e., the set of unevaluated incident edges. In initializing \(E_i\) in such a way, incident edges \((i, j)\) are considered only when the neighbor \(j \in N_i\) has not yet been a leader, as edges incident to a previous leader \(j\) have already been checked. This guarantees that network edges are considered only once for quadrupling and pebble covering. Also, note that each leader receives the current size of the independent edge set \(|E^*(t)|\) in initialization, by embedding \(|E^*(t)|\) in the leadership auction. This allows a leader to terminate the algorithm when \(2n-3\) independent edges have been identified.

The leader executes the procedure \textsc{LeaderRun} depicted in Algorithm 2 given in the Appendix, to accomplish the task of evaluating its incident edges. First, recall that in checking independence a pebble covering for each quadrupled edge \(e_i \in E_i\) must be determined. As the pebble information is distributed across the network, the lead agent must therefore request pebbles through messaging in an attempt to assign pebbles to \(e_i\). After making a pebble request, the lead agent then pauses execution and waits for pebble responses before continuing; a method often referred to as blocking.

When there exists no unfulfilled pebble requests, the lead agent starts or resumes the quadrupling procedure on the current incident edge \(e_i \in E_i\), lines 3–11. For each copy of \(e_i\), the leader searches for a pebble to cover \(e_i\), first by looking locally for free pebbles, assigning \(e_i\) to \(P_t^i\), if found. If no local pebbles are available, the agent then sends a \textsc{PebbleRequestMsg} to the endpoint of the first edge to which a pebble is assigned, requesting a free pebble. If a pebble is received from this request, the quadrupling process continues, otherwise another request is sent to the endpoint of the second edge to which a pebble is assigned. In sending requests only along \((i, j) \in P_i\), we properly evaluate independence with respect to \(E^*\), as each \((i, j) \in E^*\) must have an assigned pebble from previous evaluations of independence.

As established by Lemma 2.2, the outcome of the quadrupling process, i.e., the existence of 4 free pebbles in the network, dictates the independence of edge \(e_i\). If the leader fails to receive 4 pebbles to cover \(e_i\), the edge is deemed redundant and evaluation moves to the next member of \(E_i\). On the other hand, for any edge \(e_i\) with a pebble covering, obtained through a combination of local assignment and pebble responses, the following actions are taken, lines 13–24. First, we return 3 pebbles to the endpoints of \(e_i\), leaving a single pebble on \(e_i\) to establish independence, and then add \(e_i\) to \(E^*_i\). If in adding \(e_i\), \(2n-3\) independent edges have been identified, the leader sends a simple message to the network indicating that the graph is rigid, and the algorithm terminates. Otherwise, the leader moves to the next member of \(E_i\) and begins a new quadrupling process. When all members of \(E_i\) have been evaluated, the leader initiates the auction (4) to elect the next leader. The process of edge quadrupling, pebble requests, edge covering, and expansion of the independent set then continues from leader to leader until either the network is found to be rigid, or every agent has been a leader, indicating non-rigidity.

**C. Inter-Agent Messaging**

As each leader attempts to expand \(E^*_i\) through quadrupling each of its members, free pebbles are needed to establish a pebble covering. We facilitate the pebble search by defining asynchronous message \textsc{PebbleRequestMsg}, accompanied by response messages \textsc{PebbleFoundMsg} and \textsc{PebbleNotFoundMsg}, indicating the existence of free pebbles. The arrival of these messages then triggers message handlers that form the foundation of the pebble search mecha-
nism. For technical details of the protocol, see the pseudocode given in the Appendix.

The reception of a PebbleRequestMsg initiates the handler HANDLEPEBBLEREQUEST depicted in Algorithm 2. Each pebble request is marked with a unique identifier, originating from the lead agent, defining the pebble search to which the request is a member and ensuring proper message flow in the network, lines 2–5. For unique requests, the receiving agent first attempts to assign local pebbles to the edge connecting the pebble requester, i.e., a pebble shift operation, lines 7–9. If a free pebble is available for the shift, a PebbleFoundMsg is sent in response, allowing the requester to free an assigned pebble for either local assignment or to itself respond to a pebble request. If instead the request recipient has no free pebbles, the agent forwards the request to the endpoints of its assigned pebbles, recording the original pebble requester such that responses can be properly returned, lines 11–12. Notice that this messaging logic not only facilitates the pebble shift and assignment rules of the original pebble game, but also eliminates the need for explicit message routing. Instead, it is previous pebble assignments that dictate message routing.

When the PebbleFoundMsg response to a pebble request is received it triggers the handler HANDLEPEBBLEFOUND depicted in Algorithm 3. Similar to the shifting action of HANDLEPEBBLEREQUEST, the agent first frees the local pebble assigned to the edge connecting the responder, line 2, and then uses the newly freed pebble depending on leader status. If the agent is currently the leader, line 4, the freed pebble is assigned locally to \( e_i \), continuing the edge quadrupling process and relieving the request blocking condition. For non-lead agents, a pebble shift is performed to again free a pebble for a requesting agent, indicating the shift by returning a PebbleFoundMsg to the requester, lines 6–7.

Finally, the PebbleNotFoundMsg response to a pebble request initiates the handler HANDLEPEBBLENOTFOUND depicted in Algorithm 5. For both leaders and non-leaders, the lack of a free pebble initiates a further search in the network, along untraversed incident edges to which a pebble is assigned, line 3. However, if both available search paths have been exhausted, the leadership status of the receiver dictates the action taken. In the case of a non-leader, line 11, the response is simply returned to the original requester in order to initiate further search rooted from the requester. For a leader, lines 6–9, the lack of free pebbles in the network indicates precisely that the conditions of Lemma 2.1 do not hold, implying the currently considered edge \( e_i \) is redundant. The edge \( e_i \) is removed from consideration by returning all pebbles assigned during the covering attempt to the endpoints of \( e_i \), and the process is moved to the next incident edge. A basic illustration of a snapshot of the \( S \) algorithm is given in Fig. 4.

D. Complexity Analysis

The complexity of \( S \) is promising for realistic decentralized operation:

**Proposition 3.2** (S complexity): By construction, executions of the \( S \) algorithm have worst-case \( O(n^2) \) messaging complexity and \( O(n) \) storage scaling.

**Proof:** As the pebble game exhibits \( O(n^2) \) complexity \([28]\), our pebble messaging scales like \( O(n^2) \). In applying leader auction \( 4 \) we incur \( O(n^2) \) as we expend \( O(n) \) auction messaging for \( O(n) \) leaders. Equivalently, the centralized execution takes \( O(n^2) \) and we simply apply an \( O(n^2) \) decentralization to provide the algorithm with the appropriate runtime information. Thus, our overall algorithm will run with \( O(n^2) \) complexity. Finally, the per-agent storage complexity scales like \( O(n) \), the maximal cardinality of \( N_i \), as assignments to \( E_i \) occur over only edges incident to \( i \), line 23 Algorithm 2.

The above result demonstrates that our proposed \( S \) algorithm represents a fully decentralized and efficient solution to the planar generic rigidity evaluation problem, providing the opportunity to exploit the vast advantages of network rigidity in realistic robotic networks. Technical analysis and detailed pseudocode for the \( S \) algorithm can be found in the Appendix.

IV. EXPLOITING STRUCTURE TOWARDS PARALLELIZATION

To fully exploit a distributed multi-agent system, we seek a parallelization of the algorithm proposed in Section III with the goal of reducing the overall execution time of rigidity evaluation, and rendering real-world application feasible. It turns out that evaluating network rigidity is intrinsically serial and centralized in nature, making it difficult to asymptotically reduce the computational complexity through parallelization. Instead, we aim to provide a parallelization that is advantageous under realistic circumstances, yielding both non-trivial runtime improvements and uses for building rigid networks, with no additional hardware or communication requirements. At a high level, our scheme consists of identifying local edge addition operations that preserve independence, and allowing the agents to apply these rules simultaneously to build a set of independent edges. We will develop these ideas in the sequel and direct the reader to Remark 4.2 for a complete summary of the advantages of parallelization.

A. Independence Preserving Operations

Let us begin by formally defining addition and subtraction operations for graph edges as follows.

**Definition 4.1** (Edge addition/subtraction [10]): Consider a graph \( G = (\mathcal{V}, \mathcal{E}) \) and let the graph augmented with an
edge $e$ be denoted $G^+ = (\mathcal{V}, \mathcal{E} \cup \{e\})$. Similarly, the graph $G$ with $e$ removed is denoted by $G^- = (\mathcal{V}, \mathcal{E} \setminus \{e\})$. We refer to the operation $[\cdot]_e^+$ such that $G^+ = [G]_e^+$ as edge addition. Likewise, the operation $[\cdot]_e^-$ such that $G^- = [G]_e^-$ as edge subtraction (or deletion).

Now we are prepared to consider independence preserving graph operations. Specifically:

**Definition 4.2 (Independence preserving operations):** We call the edge operations $[\cdot]_e^+$ and $[\cdot]_e^-$ over graph $G = (\mathcal{V}, \mathcal{E})$ having independent edges $\mathcal{E}$ satisfying Definition 2.2 independence preserving (IP) if $\mathcal{E} \cup \{e\}$ and $\mathcal{E} \setminus \{e\}$ are themselves independent, respectively. Clearly, all operations $[\cdot]_e^+$ over independent edges $\mathcal{E}$ are independence preserving. Also, we have that addition operations $[\cdot]_e^+$ that are not independence preserving imply $e$ is redundant with respect to $\mathcal{E}$.

Thus, we seek IP edge addition operations that enable the construction of $\mathcal{E}^+ \in \mathcal{E}^*$ in a parallel fashion. Then, given an initial independent set $\mathcal{E}^+(0) = \emptyset$ with associated graph $G^+ = (\mathcal{V}, \mathcal{E}^*)$, we can generate the sequence

$$G^+(0) = (\mathcal{V}, \emptyset) \quad G(k) = [G(k-1)]_e^+, \quad k = 1, \ldots, m \quad (6)$$

where if each $[\cdot]_e^+$ is independence preserving, the resulting graph $G^+(m)$ possesses independent edges. Then, if each operation $[\cdot]_e^+$ is local to the endpoints of edge $e$, sequence (6) can be achieved in parallel.

To identify such operations, we take inspiration from the Henneberg construction, a sequence of node and edge additions that iteratively builds a minimally rigid graph [27]. First, consider a simple rule based on the membership of $v_i$ or $v_j$ as endpoints in $G^+(k)$:

**Definition 4.3 (Endpoint expansion rule):** Consider the graph $G = (\mathcal{V}, \mathcal{E})$ and the associated node set

$$G^+ = \{v_i \in \mathcal{V} \mid \exists j \in I, (i, j) \in \mathcal{E}\},$$

containing the nodes in $\mathcal{V}$ which are endpoints of edges in $\mathcal{E}$. Define the endpoint expansion rule (EER) as the edge addition operation $[G]^+ = [G]_e^+$ possessing the property that $|V|^+ > |V|$, where $|V|^+$ are the endpoints of $G^+$. Notice that for an EER operation it trivially holds that $2 \geq |V|^+ - |V| \geq 1$.

Clearly the endpoint expansion rule [43] is limiting in terms of the identified set $\mathcal{E}^*$, specifically as the identified set can be described as the union of spanning trees over $G$, a direct consequence of expanding endpoints in a graph, as in Fig. 5a. Thus, we can further consider a two edge rule that is also independence preserving:

**Definition 4.4 (Two incident edge rule):** Consider the graph $G = (\mathcal{V}, \mathcal{E})$ and the augmented graph $G^+$ through addition of edge $e \subset (i, j)$. The two incident edge rule (TIER) is an edge addition operation $[G]^+$ where it holds that $(N^+_i \leq 2) \lor (N^+_j \leq 2)$ over $G^+$, with $(N^+_i \geq 1) \land (N^+_j \geq 1)$ over $G$, otherwise the edge operation would constitute an endpoint expansion.

We illustrate the rules of Definitions 4.3 and 4.4 in Figs. 5a and 5b, respectively, with the independence preservation of the proposed rules given by Proposition A.4 in the Appendix. The EER and TIER operations indicate first that an independent set could be built incrementally as in (6), much like the original pebble game. However, instead of requiring inherently global pebble searches, the EER and TIER operations are distinctly local in nature, making them amendable to parallel implementation.

B. Gossip-like Messaging for Parallelization

We now take inspiration from the randomized communication scheme typical of gossip algorithms, e.g., [41], [42], to define the execution and messaging structure for partial parallelization of rigidity evaluation. Each agent $i$ exchanges inter-neighbor messages in an attempt to assign incident edges $(i, j), \forall j \in N_i$ to $\mathcal{E}^+_i$ according to the EER and TIER rules. Such a construction, denoted as algorithm $P$, allows the network to determine a subset of independent edges $\mathcal{E}^+_P \subseteq \mathcal{E}^*$ with significantly reduced execution time and messaging, as will be verified in Section V-B. The technical pseudocode for our parallelization is given in the Appendix.
independence set $E_i^*$ by exploiting messaging to determine a neighbor $j$'s feasibility as an endpoint in $G^*(k)$, ensuring that the EER and TIER conditions are fulfilled. Thus, for each edge $e_i \triangleq (i,j)$ chosen randomly from $E_i$, agent $i$ requests neighbor $j$'s commitment to $G^*$ through message EDGE_REQUEST_Msg$(i,j)$, shown in Algorithm 6 lines 3-4, and blocks further edge consideration, similar to the leader logic described in Section III-B. When all edges in $E_i$ have been considered, line 8, the agent enters into the idle state until the network stopping condition is met. Algorithm termination then occurs when all agents have entered into the idle state. The remaining logic for independent edge selection and rule checking resides in the message handlers related to EDGE_REQUEST_Msg$(i,j)$ of Algorithm 7 as will be discussed in the following.

C. Inter-Agent Messaging

The reception of an edge request message by agent $i$ from neighbor $j \in N_i$ triggers the message handling logic HANDLE_EDGE_REQUEST$(i,j)$ depicted in Algorithm 6. Recall that when an agent $i$ receives such a request from $j$ it indicates an attempt by $j$ to make the assignment of edge $(j,i)$ to its independent set $E_j^*$. Requests for an edge assignment must therefore be tested for fulfillment of the EER and TIER rules. However, as all agents are trying to add incident edges to the independent edge set simultaneously, it may be the case that two agents sharing an edge conflict on which agent takes the edge, specifically as both cannot. Thus, receiving agent $i$ first determines whether a request represents a conflict over edge $(i,j)$. If there is a conflict, agents $i$ and $j$ resolve the conflict as follows:

Remark 4.1 (Conflict resolution): We define an edge conflict in the execution of Algorithm $P$ as the state in which an agent $i \in I$ has received an EDGE_REQUEST_Msg$(i,j)$ from $j \in N_i$, where for $i$ it holds that requestedFrom$(i) = j$. Intuitively, this state indicates that both $i$ and $j$ are attempting to add edge $(i,j)$ to $E^*$, a condition that would introduce inconsistencies in $|E^*|$. In such scenarios we assume there exists a function RESOLVE_CONFLICT$(i,j): E \rightarrow I$ indicating the conflict winner, such that

$$\text{RESOLVE_CONFLICT}(i,j) = \text{RESOLVE_CONFLICT}(j,i) \quad (8)$$

for all $i \neq j \in I$. Now, resolving a conflict is simply a matter of first deciding which agent wins and receives the edge, and then ensuring that the losing agent does not take the edge. The winner of a conflict is decided according to the predetermined policy, RESOLVE_CONFLICT, which is chosen as a design parameter, and the loser is denied the edge by rejecting its edge request, lines 2-8. This is accomplished by saturating the commitments of the winning agent, such that the edge request of the losing agent cannot be fulfilled due to violation of the EER and TIER rules. In achieving condition $5$, we could consider various options, e.g., balancing $|E_i^*|$ and $|E_j^*|$ in choosing the winner, applying a simple predetermined condition such as agent label, or perhaps a more complex method such as considering optimal assignments based on some cost or utility function over $(i,j)$.

After properly handling potential conflicts over $(i,j)$, receiver $i$ then simply responds to $j$ via EDGE_RESPONSE_Msg$(i,j,\text{response})$ indicating its current commitment to $E^*$, increases its own commitment count to guarantee independence preservation, and removes $(i,j)$ from $E_i$ to avoid double checking, lines 8-13.

In coherence with the edge request logic, in receiving an EDGE_RESPONSE_Msg$(i,j,\text{response})$, an agent $i$ acts according to Algorithm 8. Again, as an edge response conveys agent $j$’s commitment to $E^*$, it is validated against the EER and TIER rules, with success resulting in the assignment of $(i,j)$ to $E_i$ and an incrementing of committed$(i)$ as $i$ is now an endpoint of $G^*$, lines 3-4. After independence preserving assignment, agent $i$ simply moves to consider its next incident edge by choosing a random $(i,j) \in E_i$, lines 7-8.

At the conclusion of algorithm $P$, a distributed independent edge set $E_P^*$ has been computed across each $E_i^*$. However, by the EER and TIER definitions, it must hold generally that $E_P^* \subseteq E^*$. Thus, to fully determine $E^*$ we generate a composite algorithm by passing the terminal state of $P$ to the serialized algorithm $S$ in order to apply global network information in completing $E^*$. The initial conditions for the serialized algorithm are generated from the output of the parallel algorithm in a way that it is feasible that the serial algorithm itself had run up to that point. Thus, the output of the parallel algorithm must be shaped to mimic the conditions of the serialized algorithm. To do so we simply apply a network summing algorithm to determine the overall size of the independent edge set (i.e., the sum of the local set sizes), and then assign pebbles to cover the independent edges as required for the pebble game. Specifically, we first apply gossip averaging $[41]$, to yield $|E_P^*|$. Then, local pebble assignments can easily be determined locally, as verified.
by Proposition A.8. This composite construction, which we will denote $\mathcal{P} + \mathcal{S}$, is illustrated in Fig. [3]. An example execution of the parallel algorithm is depicted in Fig. [4].

D. Complexity Analysis

Towards the real-world applicability of our propositions, we have complexity that scales well with network size.

Proposition 4.1 ($\mathcal{P} + \mathcal{S}$ complexity): By construction, executions of the $\mathcal{P} + \mathcal{S}$ algorithm have worst-case $O(n^2)$ messaging complexity, and $O(n)$ storage scaling.

Proof: For the $\mathcal{P}$ portion of the $\mathcal{P} + \mathcal{S}$ algorithm, we have that each of $n$ agents communicates with at most $n - 1$ neighbors, yielding $O(n^2)$ messaging. The gossip averaging for determining $|E^*_P|$ exhibits $O(n \log n)$ messaging [41], while a local pebble assignment trivially requires $O(n^2)$ operations. Thus we have an overall worst-case message complexity of $O(n^2)$ for a $\mathcal{P} + \mathcal{S}$ as $\mathcal{S}$ also has $O(n^2)$ messaging by Proposition 3.2. The overall storage complexity follows directly from the fact that assignments to $E^*$ can be made only locally by each agent $i \in \mathcal{I}$, and thus scales like $O(n)$.

Finally, to close we can also roughly evaluate the expected improvement provided by the partial parallelization due to $\mathcal{P}$:

Proposition 4.2 (Parallel identification): An execution of $\mathcal{P}$, applied to a graph $G = (V, E)$ (where we assume every $v_i \in V$ is an endpoint in $E^*$), with $n \geq 3$ must result in the terminal condition

$$|E^*_P(i_P)| \geq \left\lceil \sum_{i} E^*_P(i_P) \right\rceil \geq \left\lceil \frac{n - 1}{2} \right\rceil + 1 \quad (9)$$

where $\lceil \cdot \rceil$ is the standard ceiling operator, yielding a lower bound on the independent edges identified by the $\mathcal{P}$ algorithm.

Proof: First, notice that we disregard the case of $n = 1$ (as $|E| = 0$), and for $n = 2$ we can always identify the single member of $E^*$ due to symmetric conflict resolution. Now, observing that the single $n = 2$ graph is a worst-case in terms of detectable independent edges, with $|E^*| = [n/2]$, for $n \geq 3$ we can construct a similar worst-case graph by appending a single node and edge to the $n - 1$ worst-case, as the appended edge will always be detectable by an EER operation. As we add a single detectable edge to the previous worst-case, a simple inductive argument yields the result.

The above result states directly that we can always detect a spanning tree over $G$ using our proposed parallel and distributed interactions. Further, the number of edges that the parallelization identifies directly affects the number of edges left to be found, and thus the complexity of the serial execution. A summary of the advantages of our parallelization are given in the following remark, while technical analysis and detailed pseudocode can be found in the Appendix.

Remark 4.2 (Parallelization Benefits): First, as the parallel algorithm identifies at least a spanning tree in its execution by Proposition 4.2, it can quickly help to determine when the network is non-rigid without having to run the serialized step, yielding significant speed advantages in those scenarios. Additionally, as the parallel algorithm takes advantage of independence preserving operations, one could leverage it to build a rigid network autonomously and efficiently, e.g., in constructing a localizable sensor network topology. From a practical perspective, the parallelization is simply a more intelligent use of available network resources, i.e., it can be implemented without any additional communication or hardware requirements, so even a factor of two speedup may be convenient in practice. Also, such speedups relate well to realistic scenarios such as the time scales of external network influences and the speed of rigidity evaluation (see Section V-C). Even constant factor speedups can expand the applicability of our algorithms under faster switching topologies or environmental conditions.

V. Simulation Results

A. A Rigidity Control Scenario

We wish to demonstrate here a scenario where network rigidity can be controlled in a dynamic multi-robot system. To begin, consider that in controlling generic rigidity we need only to disallow the loss of independent edges $(i, j) \in E^*$. For this purpose, we can employ the constrained interaction framework proposed by Williams and Sukhatme in [4], a very brief overview of which will be given here. Assuming proximity-limited sensing and communication, together with agent dynamics $\dot{x}_i = u_i$, the constrained interaction framework regulates link addition and deletion spatially, through a switching combination of hysteresis, attraction, and repulsion, in order to satisfy a desired set of constraints. In particular, each agent is assigned predicates $P_{ij}^0, P_{ij}^d : V \times V \leftrightarrow \{0, 1\}$ that indicate constraint violations if link $(i, j)$ were gained or lost. Thus, in applying our proposed decentralized pebble game...

![Fig. 7. Parallel messaging for a minimally rigid graph with $n = 4$. Inter-agent requests are denoted by solid arrows, responses by dashed arrows, and pebble assignments by solid dots. Notice that conflicts occur in (a) over $(v_1, v_2)$ and in (c) over $(v_2, v_4)$, with resolution dictated by agent label $i < j$ for $j \in \mathcal{N}_i$.](image-url)
to identify local sets \( E^*_i \forall i \in I \), we arrive directly at rigidity preserving predicates. That is,

\[
P^a_{ij} \triangleq 0, \quad P^d_{ij} \triangleq (i, j) \in (E^*_i \cup E^*_j)
\]  

(10)

Now, we are prepared to present our rigidity control simulation results. We assume a system of \( n = 9 \) mobile agents, each with proximity-limited communication and sensing, applying for the sake of link deletion, a dispersive objective controller, yielding agent controllers with generic form:

\[
u_i = u_{C|I} - \nabla x_i \left( \sum_{i \in N_i} \frac{1}{\|x_i - x_j\|^2} \right) \quad \forall i \in I
\]  

(11)

where \( u_{C|I} \) is the control contribution due to the constrained interaction framework and predicates (10). The agents begin in the fully connected initial configuration given by the ring network depicted in Fig. 8a, satisfying the initial condition \( G(0) \in G_R \). Through controllers (11), the agents reach intermediate configurations given by Figs. 8b and 8c, ultimately terminating in the final configuration in Fig. 8d. Fig. 9 depicts the spatial size of the swarm, i.e., the largest distance between any two agents, and the size of the independent edge set, which dictates network rigidity. Thus Fig. 9 demonstrates that the dispersive objective is achieved through increasing swarm size, and that the network remains rigid as the size of the independent set is bounded below by \( 2n - 3 \).

B. Contiki Implementation: Real-World Feasibility Results

To determine the performance of our algorithms under realistic networking conditions, we consider the Contiki operating system, together with the Cooja network simulator. We implemented both the serial and parallel decentralized pebble game for Contiki and tested our codebase against a range of emulated hardware platforms and communication stacks for correctness. A Monte Carlo set was simulated by generating rigid and non-rigid networks for \( n \in \{5, 29\} \), yielding the results depicted in Fig. 10. Specifically, Fig. 10 (top) compares the execution time (seconds) for the serial and parallel algorithms, while Fig. 10 (bottom) shows the per-agent messaging burden. It is clear that both of our algorithms exhibit feasible and efficient performance, with actual scaling that is approximately \( O(n) \) in both execution and messaging. The parallel version however represents our goal of real-world capability by outperforming the serial version by a factor 2, as even in reasonably sized networks, execution times for evaluation are under 1 second. An initial version of the base code for our proposed algorithms has also been released for application in the robotics community.

C. Realistic Considerations

Realistic applications present difficult and unpredictable influences, e.g., wind gusts or variability in ocean currents in autonomous surface vehicles (ASVs). These environmental variables and their timescales will directly impact the capability of the network to determine network rigidity in a timely fashion, as the network topology may change too often due to uncontrollable influences. Thus, the relationship between the parameters of an application, the properties of the employed communication network, and the numerical bounds on rigidity evaluation run-time must all be well understood by an implementer. Informally, the switching time of the topology, which is dictated by communication hardware and application-specific factors, and the network size become crucial design variables as they determine the feasibility of computing network rigidity in time to steer the network away from non-rigidity. In general, networks which exhibit short switching times will...
require smaller network size or faster communication to combat external influences for feasible operation.

VI. CONCLUSIONS AND FUTURE WORK

In this paper, we considered the problem of evaluating the rigidity of a planar network under the requirements of decentralization, asynchronicity, and parallelization. We proposed the decentralization of the pebble game algorithm of Jacobs et al., based on asynchronous inter-agent message-passing and distributed auctions for electing network leaders. Further, we provided a parallelization of our methods that yielded significantly reduced execution time and messaging burden. Finally, we provided a simulated application in decentralized rigidity control, and Monte Carlo analysis of our algorithms in a Contiki networking environment, illustrating the real-world applicability of our methods.

Directions for future work include demonstrating our methods on robotic hardware in field experiments, extensions to global rigidity for even stronger guarantees in multi-robot applications, and the inclusion of network utility metrics to yield decentralized rigidity evaluation and control with provable optimality conditions.

APPENDIX

A. The Serialized Algorithm

To complement our discussion of the $S$ algorithm, we analyze the correctness, finite termination, and cost properties of our algorithms. First, we formally establish the stopping condition for the $S$ algorithm:

**Definition A.1 ($S$ stopping condition):** As previously discussed, the $S$ algorithm terminates upon satisfaction of the following condition:

$$f_{\text{stop}}^S \triangleq \left\{ \left( \sum_{i=1}^{n} b_i = 0 \right) \lor \left( \sum_{i=1}^{n} |E_i^*| = 2n - 3 \right) \right\} (12)$$

where the $\sum_i b_i = 0$ indicates that all agents have been a leader, and $\sum_i |E_i^*| = 2n - 3$ is detected by the lead agent on line 24 of Algorithm 2.

Next, we verify that our formulation guarantees the entire network is evaluated for rigidity, with no edge reconsideration, and further that mutual exclusion of the local independent sets holds:

**Proposition A.1 (Edge consideration, mutual exclusion):** Disregarding algorithm termination when $|E^*| = 2n - 3$, every $(i, j) \in E$ is eligible to be considered for independence. Further, $E_i^* \cap E_j^* = \emptyset$ holds for all $i \neq j$.

**Proof:** These results are a simple consequence of the guaranteed convergence of auction (4), $b_i = 0$ for all $\text{beenLeader}(i) = 1$ guaranteeing no re-election, and the initialization of $E_i$ with edges not shared with previous leaders.

To ensure timely results, we must also have finite termination of $S$:

**Proposition A.2 ($S$ termination):** Consider the execution of the $S$ algorithm as described in Sections III-A, III-B, and III-C. By construction, it follows that the stopping condition $f_{\text{stop}}^S$ of (12) is satisfied after a finite number of clock ticks.

**Proof:** We can guarantee no message-induced race conditions by Assumptions 2 and 3, and that there exist no algorithmic race conditions due to the internal blocking on line 2 of Algorithm 2. From the request checking mechanism (line 2 Algorithm 3) and the guaranteed delivery of inter-agent messages by the best-effort Assumption 3, we have that all pebble request messaging rooted at agent $i$ with $\text{isLeader}(i) = 1$ is finite, i.e., every pebble request receives a response. Now, the finiteness of execution is a direct consequence of the finiteness of each $E_i, \forall i \in \mathcal{I}$ and the finite convergence of auction (4) [40], as there exists no leader re-election by construction.

Now we come to our primary result concerning the correctness of the $S$ algorithm:

**Proposition A.3 ($S$ correctness):** Consider an execution of $S$, applied to a graph $G = (\mathcal{V}, \mathcal{E})$. It follows that by construction we are guaranteed to identify $|E^*| = 2n - 3$ independent edges when $\mathcal{G} \in \mathcal{G}_R$, and $|E^*| < 2n - 3$ otherwise, i.e., $S$ properly identifies the generic rigidity of $\mathcal{G}$.

**Proof:** First, notice that by Proposition A.1 we can ensure that every $(i, j) \in E$ is eligible for quadrupling and pebble covering as dictated by the original pebble game [28], and further that $E_i^* \cap E_j^* = \emptyset$ holds for all $i \neq j \in \mathcal{I}$ ensures that $|E^*|$ is properly tracked by our distributed storage. Thus, correctness is shown by arguing that our leadership and messaging formulation is faithful to the rules of the pebble game. This result follows by observing that pebble assignments and shift operations are only made locally (line 8 of Algorithm 2) line 7 of Algorithm 3 and lines 2, 4, and 8 of Algorithm 3, and that the pebble search mechanism respects the network’s distributed pebble assignments $P_i, \forall i \in \mathcal{I}$ (line 12 of Algorithm
Algorithm 2 Leader execution logic.

1: procedure LEADERRUN(i)
2: while \( e_i \triangleq (i, j) \) do \( \triangleright \) Continue pebble covering
3: while Quadrupled Copies \( \leq 4 \) do
4: if \( p_i > 0 \) then \( \triangleright \) Assign local pebble
5: \( \mathcal{P}_i \leftarrow \mathcal{P}_i \cup e_i \)
6: \( p_i \leftarrow p_i - 1 \)
7: else \( \triangleright \) Request pebble along first edge
8: \( \text{PEBBLEREQUESTMSG}(i, \mathcal{P}_i(1, 2)) \)
9: return
10: end if
11: end while
12: \( \triangleright \) Quadrupling success, return pebbles:
13: \( \mathcal{P}_i \leftarrow \emptyset \)
14: \( p_i \leftarrow 2 \)
15: Return 1 pebble to \( v_j \)
16: \( \triangleright \) Add independent edge and check rigidity:
17: \( \mathcal{E}_i \leftarrow \mathcal{E}_i \cup e_i \)
18: if \( |\mathcal{E}^*| = 2n - 3 \) then
19: Send network rigidity notification
20: return
21: end if
22: \( \triangleright \) Go to next incident edge:
23: \( e_i \leftarrow (i, j) \in \mathcal{E}_i \)
24: end while
25: \( \triangleright \) All local edges checked:
26: end procedure

Algorithm 3 Pebble request handler for agent \( i \).

1: procedure HANDLEPEBBLEREQUEST(from, i)
2: if Request Not Unique then \( \triangleright \) Already requested
3: \( \text{PEBBLENOTFOUNDMSG}(i, \text{from}) \)
4: return
5: end if
6: if \( p_i > 0 \) then \( \triangleright \) Local pebble available
7: \( \mathcal{P}_i \leftarrow \mathcal{P}_i \cup (i, \text{from}) \)
8: \( p_i \leftarrow p_i - 1 \)
9: \( \text{PEBBLEFOUNDMSG}(i, \text{from}) \)
10: else \( \triangleright \) Request along first assigned edge
11: \( \text{PEBBLEREQUESTMSG}(i, \mathcal{P}_i(1, 2)) \)
12: requester(i) \( \leftarrow \) from
13: end if
14: end procedure

Algorithm 4 Pebble found handler for agent \( i \).

1: procedure HANDLEPEBBLEFOUND(from, i)
2: if Paths Searched \( < 2 \) then \( \triangleright \) Search other path
3: \( \text{PEBBLEREQUESTMSG}(i, \mathcal{P}_i(2, 2)) \)
4: else \( \triangleright \) Search failed, no free pebbles
5: if isLeader(i) then \( \triangleright e_i \) is redundant
6: Return pebbles assigned to \( e_i \)
7: \( \triangleright \) Go to next incident edge:
8: \( e_i \leftarrow (i, j) \in \mathcal{E}_i \)
9: \( e_i \leftarrow (i, j) \in \mathcal{E}_i \)
10: else
11: \( \text{PEBBLENOTFOUNDMSG}(i, \text{requester}(i)) \)
12: end if
13: end if
14: end procedure

B. The Parallelized Algorithm

We begin with a proof of the independence preservation of the EER and TIER graph operations:

Algorithm 5 Pebble not found handler for agent \( i \).

1: procedure HANDLEPEBBLENOTFOUND(from, i)
2: if Paths Searched \( < 2 \) then \( \triangleright \) Search other path
3: \( \text{PEBBLEREQUESTMSG}(i, \mathcal{P}_i(2, 2)) \)
4: else \( \triangleright \) Search failed, no free pebbles
5: if isLeader(i) then \( \triangleright e_i \) is redundant
6: Return pebbles assigned to \( e_i \)
7: \( \triangleright \) Go to next incident edge:
8: \( e_i \leftarrow (i, j) \in \mathcal{E}_i \)
9: \( e_i \leftarrow (i, j) \in \mathcal{E}_i \)
10: else
11: \( \text{PEBBLENOTFOUNDMSG}(i, \text{requester}(i)) \)
12: end if
13: end if
14: end procedure

Proposition A.4 (Independence preservation): Consider the graph \( G = (\mathcal{V}, \mathcal{E}) \) having edges \( \mathcal{E} \) forming an independent set according to Definition 4.2. The edge addition operations \( \{\}^m \) over \( G \) abiding by the EER and TIER requirements of Definitions 4.3 and 4.4 are independence preserving in the sense of Definition 4.2, respectively. Further, considering a sequence of graphs \( \{G(0), \ldots, G(m)\} \) generated by

\[
G(0) = G, \quad G(k) = [G(k-1)]^+, \quad k = 1, \ldots, m
\]

over EER and TIER operations yields graph \( G^m \) having independent edges \( \mathcal{E}^m \).

Proof: First, consider the case of an EER operation over \( G \). By the independence of \( \mathcal{E} \), we have by Definition 4.2 that for every subgraph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \), \( |\mathcal{E}| \leq 2|\mathcal{V}| - 3 \). In the augmented graph edge \( e \) introduces expanded subgraphs containing \( e \) all having the property

\[
|\mathcal{E}| + 1 \leq 2|\mathcal{V}| - 3 + 1 \leq 2(|\mathcal{V}| + 1) - 3
\]
due to the node expansion property of the EER, all of which therefore abide by the independence subgraph property. The remaining subgraphs of \( G^+ \) are independent by assumption. Thus, we conclude that the EER operation according to Definition 4.3 is independence preserving.

Now, consider the application of the TIER operation over \( G \). By Definition 4.4 there must exist an endpoint of \( e \triangleq (i, j) \), indexed by \( i \in \mathcal{L} \), with exactly \( \mathcal{N}_i = 1 \) over \( G \). Thus, we can view the edges \( (i, j) \) and \( (i, k) \) with \( k \in \mathcal{N}_i \), and the node \( i \) as members of a two edge Henneberg operation, as in Section 3 of [27], e.g., adding vertex \( v_4 \) with edges (4,2) and (4,1) in

\[2\] line 11 of Algorithm 3, and line 3 of Algorithm 5. Finally, as there is only one leader active at any time, each quadrupling operation (lines 6-17, Algorithm 2) is sound with respect to the current set \( \mathcal{E}^* \), and the result follows.

The above result demonstrates that \( \mathcal{S} \) is sound in terms of planar rigidity evaluation.
Algorithm 6 Parallel execution logic for agent $i$.

1: procedure PARALLELRUN($i$)
2: \quad if $e_i \triangleq (i,j) \neq 0$ then $\triangleright$ Next incident edge
3: \quad EDGEREQUESTMSG($i,j$)
4: \quad requestedFrom($i$) $\leftarrow$ $j$
5: \quad return
6: end if
7: $\triangleright$ All local edges checked:
8: idle($i$) $\leftarrow$ Yes
9: end procedure

Algorithm 7 Parallel edge request handler for agent $i$.

1: procedure HANDLEEDGEREQUEST($i,j$)
2: \quad response $\leftarrow$ committed($i$)
3: \quad if requestedFrom($i$) $= j$ then $\triangleright$ Edge contention
4: \quad \quad if RESOLVECONFLICT($i,j$) $= i$ then
5: \quad \quad \quad response $\leftarrow$ 2 $\triangleright$ Ensure $i$ wins edge
6: \quad \quad end if
7: \quad end if
8: \quad EDGERESPONSEMSG($i,j$, response)
9: \quad if response $< 2$ then $\triangleright$ Max of 2 incident edges
10: \quad \quad committed($i$) $\leftarrow$ committed($i$) + 1
11: \quad end if
12: $\triangleright$ Do not double check ($i,j$):
13: $E_i \leftarrow E_i$ $- (i,j)$
14: end procedure

Algorithm 8 Parallel edge response handler for agent $i$.

1: procedure HANDLEEDGERESPONSE($i,j$, response)
2: \quad if response $< 2$ then $\triangleright$ Independence guaranteed
3: \quad \quad $E^*_i \leftarrow E^*_i \cup (i,j)$
4: \quad \quad committed($i$) $\leftarrow$ committed($i$) + 1
5: end if
6: $\triangleright$ Go to next incident edge:
7: $E_i \leftarrow E_i$ $- (i,j)$
8: $e_i \leftarrow (i,j) \in E_i$
9: end procedure

Fig. 5b. As the edge subtraction operation $[\cdot]_-$ is independence preserving, the graph $G^+$ described by applying the previous two edge Henneberg operation to $[G]_{i,k}$ has independent edges by Proposition 3.1 of [27]. Briefly, this result follows from the independence of $E$ and the relationships, $|E| = |E^+| - 2$ and $|V| = |V^+| - 1$. Thus, the TIER operation is also independence preserving.

Finally, the independence preservation of a sequence of EER and TIER operations is a trivial consequence of the initial independence of $E$ and the IP properties of each edge augmentation.

To complement our discussion of the $P + S$ algorithm, we analyze the correctness, finite termination, and cost properties of our algorithms. First, we verify that $E^*_i = \bigcup E^{*}_{E,i}$ has a valid distributed construction:

**Proposition A.5 (Parallel mutual exclusion):** Consider the application of the parallel $P$ algorithm to a graph $G = (V,E)$. It follows that upon termination we have mutual exclusion $E^*_i \cap E^*_j = \emptyset$, $\forall i \neq j \in I$.

**Proof:** Given the assumptions of asynchronicity in messaging and the FIFO queueing of received messages (Assumption 3) and execution devoid of race conditions (Assumption 2), the following scenarios must be considered, viewed from the instant when agent $i$ handles an EDGERESPONSEMSG from $j$, implying that $e_j = (i,j)$ and $(i,j) \notin E^*_i$ (line 2, Algorithm 8):

- $(i,j) \notin E^*_j$: We must consider two cases here, either $e_j = (i,j)$ or $e_j \neq (i,j)$. First, in the trivial case of $e_j \neq (i,j)$, it follows from reception of an EDGERESPONSEMSG from $j$, the atomic nature of execution, and line 13 of Algorithm 7 that regardless of assignment to $E^*_i$, $(i,j) \notin E^*_j$ for all execution $t > 0$. When $e_j = (i,j)$, the conflict resolution of line 3-7 in Algorithm 7 together with line 2 of Algorithm 8 ensures that simultaneous requests made over $(i,j)$ agree on assignment, specifically as by assumption RESOLVECONFLICT($i,j$) = RESOLVECONFLICT($j,i$).

- $(i,j) \in E^*_j$: Here it is implied that at some previous time agent $i$ received and responded to a EDGEREQUESTMSG from $j$. As agent $i$ being in a state of response reception over $(i,j)$ is contradictory given line 13 of Algorithm 7 it must be the case that requests over $(i,j)$ have been made in concert. However, as previously stated, the conflict resolution ensures $E^*_i \cap E^*_j = \emptyset$ in such scenarios.

Notice that due to the uniformity of execution and messaging logic across $i \in I$, the previous scenarios hold equivalently from the perspective of agent $j$, and thus for all pairs $\{i \neq j \mid (i,j) \in E\}$, and the result follows.

Of course, $E^*_i$ must also fulfill the independence requirements of Definition 2.2 as is shown below.

**Proposition A.6 (Parallel Correctness):** Consider the algorithm $P$ applied to a graph $G = (V,E)$. For all execution $t > 0$ it follows that edge addition operations, $E^*_i(t) = E^*_i(t) \cup (i,j)$ (line 2 Algorithm 8), are independence preserving and $\bigcup E^*_i$ is independent with $|\bigcup E^*_i| \leq 2n - 3$.

**Proof:** Resting again on Assumptions 2 and 3 and the uniform conflict resolution of RESOLVECONFLICT($i,j$), this result is a consequence of the commitment counting rules (lines 10 and 5 of Algorithms 7 and 8), and the condition on line 2 of Algorithm 8 that enforces the cardinality of endpoint $j$ in $\bigcup E^*_i(t)$. In particular, when $j \notin \bigcup E^*_i(t)$ (committed($j$) = 0), $E^*_i(t) = E^*_i(t) \cup (i,j)$ constitutes an EER operation (c.f. Definition 4.3), otherwise when $j \in \bigcup E^*_i(t)$ (committed($j$) = 1), $E^*_i(t) = E^*_i(t) \cup (i,j)$ constitutes a TIER operation (c.f. Definition 4.4). As sequences of EER and TIER operations preserve independence by Proposition A.4, $\bigcup E^*_i$ is independent, with $|\bigcup E^*_i| \leq 2n - 3$ following directly from the Laman conditions Theorem 2.1.

To ensure timely results, we must also have finite termination of $P + S$:

**Proposition A.7 ($P$ termination):** Consider the execution of the $P$ algorithm as described in Section 4B. By construction, it follows that the stopping condition, i.e., all agents are idle, is satisfied after a finite number of clock ticks.

**Proof:** We can again guarantee no message-induced race conditions by Assumptions 2 and 3. Thus, the finiteness of
execution is a direct consequence of the finiteness of each $E_i, \forall i \in \mathcal{I}$, the internal blocking on line 2 of Algorithm 6, the guaranteed delivery of inter-agent messages by the best-effort Assumption 3 and finally the symmetric conflict resolution of Remark 4 disallowing conflict based race conditions.

To constitute a valid initial condition for $S$, the pebble assignments applied to the terminal state $E^*_P$ of $P$ must be sound:

**Proposition A.8 (Pebble Assignments):** Consider an execution of $P$, applied to a graph $G = (V, E)$. There must exist a local pebble covering for every $(i, j) \in E^*_i \cap \mathcal{I}$, that is a local assignment of a pebble by either $i$ or $j$ to $(i, j)$.

**Proof:** In guaranteeing that such an assignment exists, we rely on the properties of the EER and TIER operations. As each operation $E^*_i(t^+) = E^*_i(t) \cup \langle (i, j) \rangle$ respects Proposition A.4 by Proposition A.6, we have that for any $(i, j) \in \bigcup E^*_i$ there must exist an endpoint $i$ or $j$ with at most two incident edges. From this endpoint we can thus always select a pebble to cover $(i, j)$ as each agent $i \in \mathcal{I}$ is initially assigned two pebbles.

Now we come to our primary result concerning the correctness and finite termination of the $P + S$ algorithm:

**Proposition A.9 ($P + S$ correctness and termination):** Consider an execution of $P + S$, applied to a graph $G = (V, E)$. It follows that the terminal state system:

$$P(t_P) \triangleq \left\{ \bigcup_i E^*_i(t_P), P, P_i \right\} \quad \forall i \in \mathcal{I}$$

is a valid initial condition for the $S$ algorithm, the execution of $P + S$ terminates after a finite number of clock ticks, and properly identifies the generic topology rigidity of $G$.

**Proof:** First, we have directly from Propositions A.2 and A.7 and the known finite convergence of gossip averaging [41], and the trivial finiteness of a local pebble assignment process, that the composite execution of $P + S$ terminates in finite time. Now, from Propositions A.5 and A.6 it follows that the state $P(t_P)$ represents a properly distributed and independent edge set, and from Proposition A.8 that there must exist pebble assignments $P$ that are local shift operations relative to Definition 2.3 and thus constitute a valid pebble covering. Thus the application of $S$ with input $P(t_P)$ is correct by Proposition A.3 and the result follows.

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