# Morphing Planar Graph Drawings with a Polynomial Number of Steps

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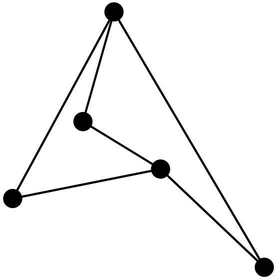
# Planar linear morphing steps of straight-line drawings

*Planar straight-line drawing of a graph*: vertices are distinct points of the plane and edges are non-intersecting straight-line segments

*Planar linear morphing step*: transformation of a planar straight-line drawing of a graph into another planar straight-line drawing of the same graph

- *moving* vertices at *constant speed* along *straight-line* trajectories
- preserving *planarity*



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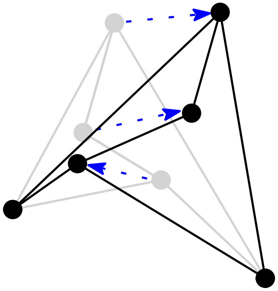

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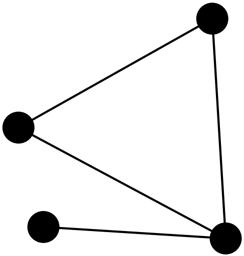
Given two planar straight-line drawings of the same graph, a *planar morphing* is a sequence of planar linear morphing steps transforming the first drawing into the second one



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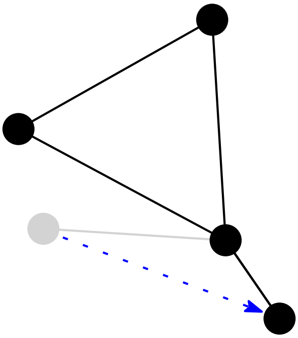

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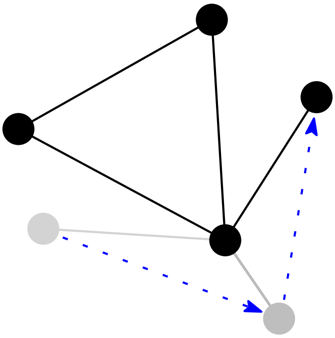

# State of the Art

- Existence of a planar morphing: $O(2^n)$ ☹
  - Between any two planar drawings of a maximal planar graph (*triangulation*) *Cairns '44*
  - Between any two planar drawings such that all faces are convex polygons (preserving the convexity in each intermediate step) *Thomassen '83*
- A polynomial number of planar linear morphing steps is guaranteed only for polygons

  *Aichholzer et al. '11*
- Related settings
  - Allowing non-linear trajectories

    *Floater & Gotsman '99, Gotsman & Surazhsky '01,'03*
  - Allowing bent edges

    orthogonal drawings *Lubiw et al. '06*

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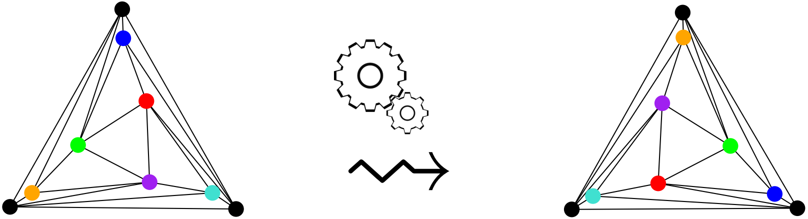
# Our result: triangulations

**Theorem**

*Given any two planar straight-line drawings of the same triangulation, there exists a planar morphing between them with $O(n^2)$ planar linear morphing steps*



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**Theorem**

*Given any two planar straight-line drawings of the same graph, there exists a planar morphing between them with $O(n^4)$ planar linear morphing steps*

- Extend the two drawings to a pair of drawings of the same triangulation
  - it can be done by adding $O(n^2)$ vertices *Aronov et al. '93*
- Apply the algorithm for triangulations

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# Preliminaries

*Kernel of a polygon* **P**: convex set $K$ of internal points of **P** having "direct visibility" to all the vertices of **P**

- If $|\mathbf{P}| \leq 5$, then $K \neq \emptyset$ and $K \cap V(\mathbf{P}) \neq \emptyset$

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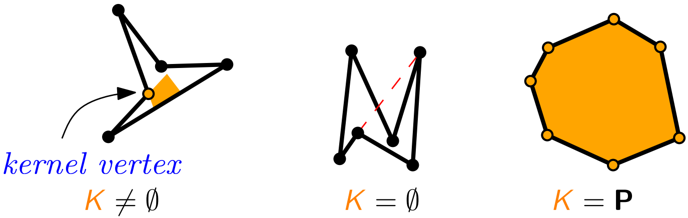

# Elementary operations

## Property (by Euler's formula)

*There exists an internal vertex $v$ whose neighbors induce a simple (without chords)* **polygon P**, *with* $|\mathbf{P}| \leq 5$

Contraction:
$v$ can be *contracted* to a *kernel-neighbor* $v'$

Actually, $v$ remains "suitably close" to $v'$ during the morphing

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If $v$ has been contracted to $v'$, $v$ can be *extracted* from $v'$ and placed on any point of the kernel of **P**

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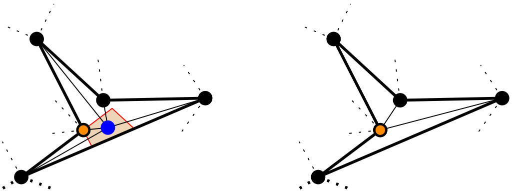

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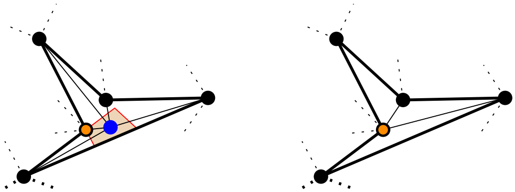

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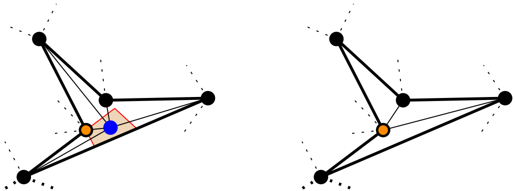

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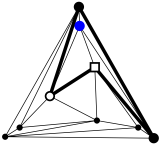

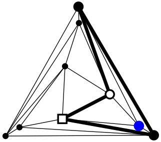

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$$T(n) = 2T(n-1) + O(1) \implies T(n) \in O(2^n)$$

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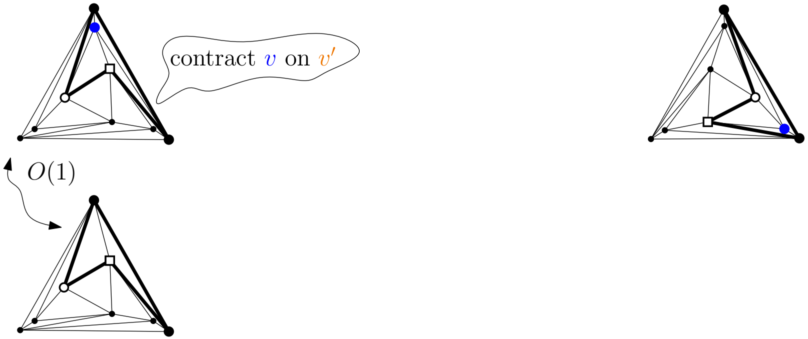


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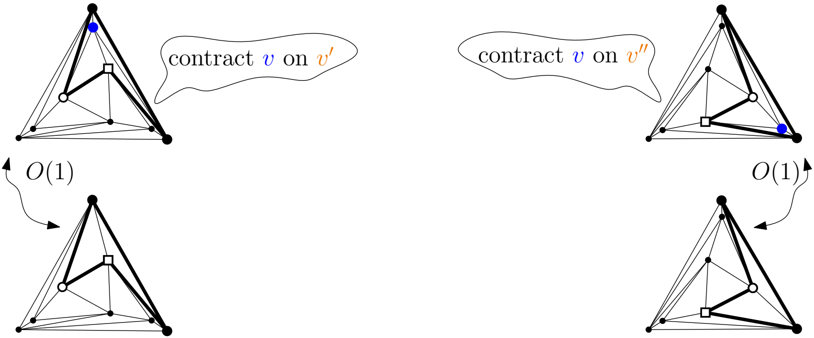


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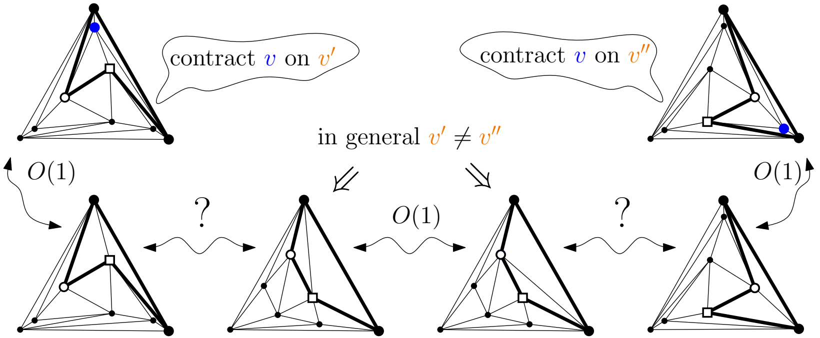


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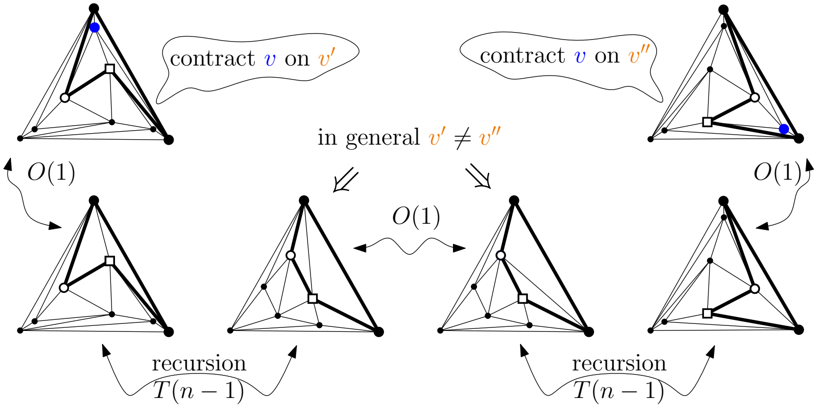


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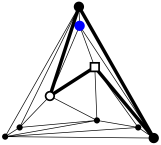

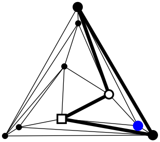

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$\implies T(n)$ polynomial if $T_{conv}(n)$ polynomial

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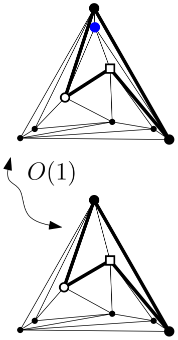


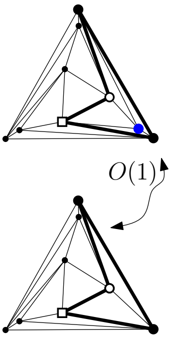


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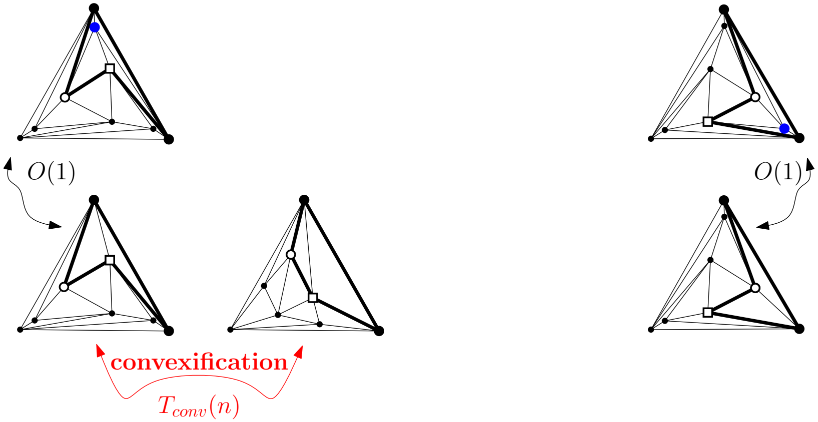


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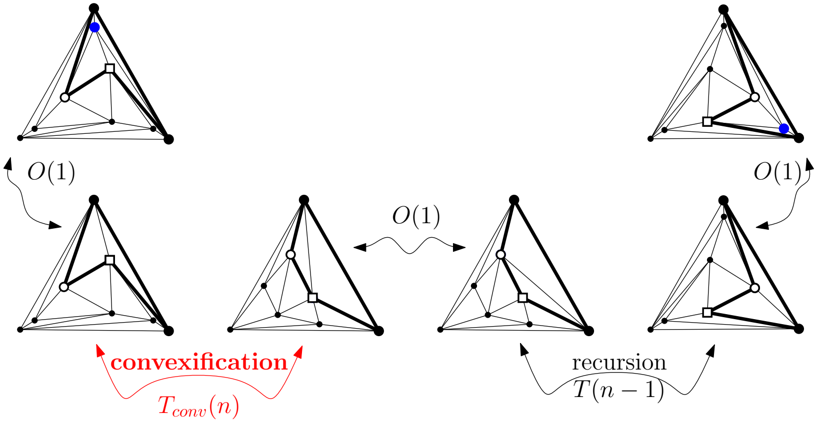


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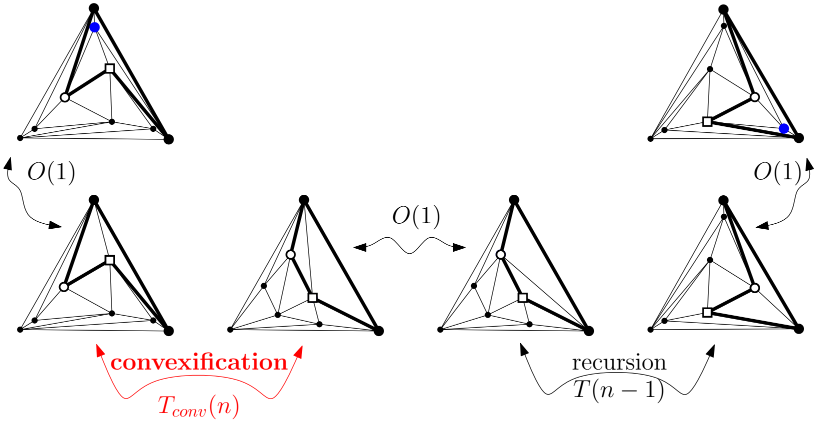


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# Computing $T_{conv}(n)$

## Problem (Convexification)

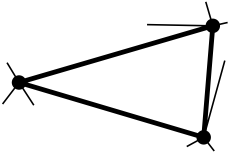
*Transform the drawing of the triangulation in such a way that vertices $v'$ and $v''$ (the kernel-neighbors of $v$ in the two contractions) become kernel-vertices of* **P** *with a polynomial number of linear morphing steps*

It can be done by:

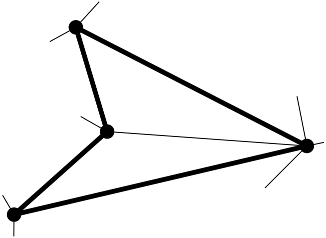
- *contracting* vertices of the graph
  - not belonging to the external face
  - without inducing external chords on **P**
- *extracting* vertices in reverse order
- applying *linear morphing steps* to handle some special cases

# Convexification of **P**

If $|\mathbf{P}| = 3$, it is already convex!



The cases where $3 < |\mathbf{P}| \leq 5$ have to be handled



# Convexification of 5-gons

The problem can be reduced to the convexification of 4-gons

## Non-adjacent kernel-neighbors

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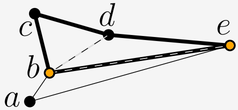


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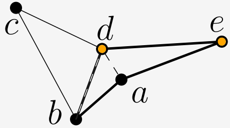

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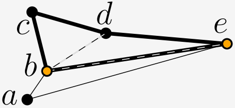


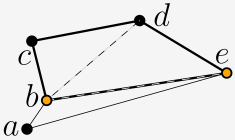


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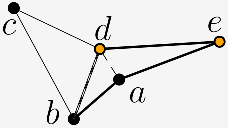

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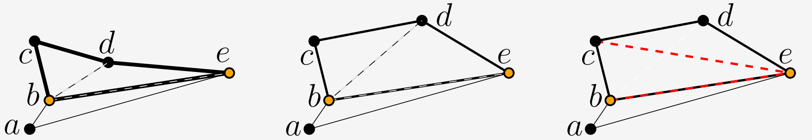


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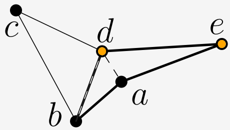

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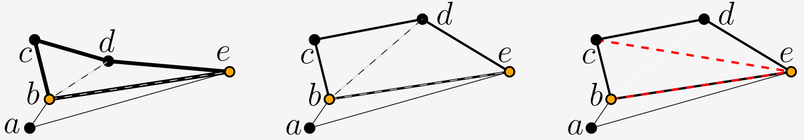


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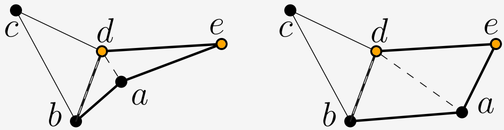

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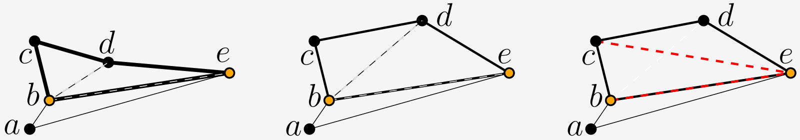


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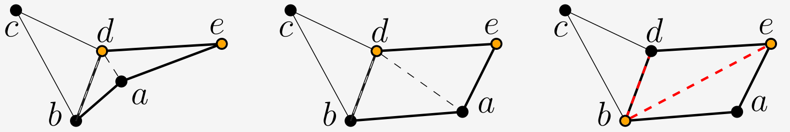
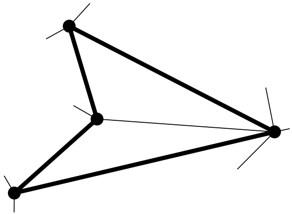


# Convexification of 4-gons

*Value* of a vertex: $val(v) = 6 - \deg(v) \implies \sum_v \text{val}(v) = 12$

A contractible vertex is *problematic* if:

- it belongs to **P** and is not on the outer face
- it is on the outer face
- its contraction would induce an *external chord* on **P**



Sometimes we can deal with problematic vertices

. . . but sometimes we cannot.

In this case the values of the problematic vertices sum up to *at most* 11 $\implies$ there exists a *non-problematic* contractible vertex.☺

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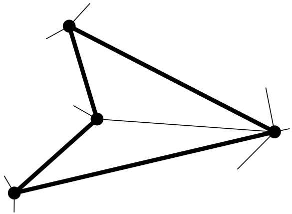

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# Dealing with problematic vertices: an example

There exist at most two chord-inducing vertices:

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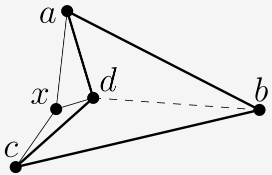


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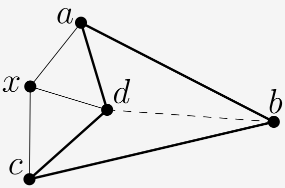

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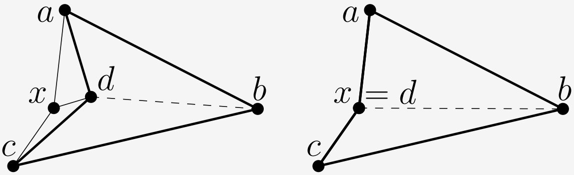
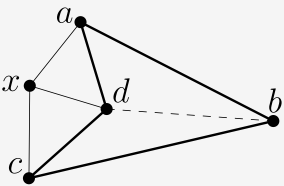


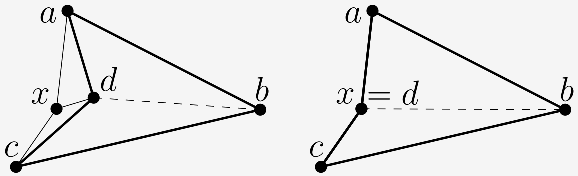

**Outside $\triangle_{abc}$**



# Dealing with problematic vertices: an example

There exist at most two chord-inducing vertices:

## Inside $\triangle_{abc}$

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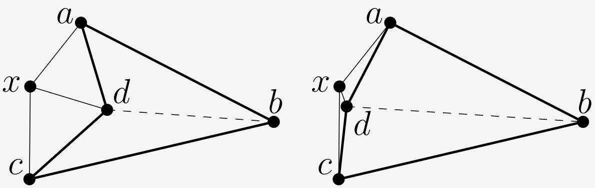

# Computational Complexity

**Total number of steps**

$T(n) = T_{conv}(n) + T(n-1) + O(1)$

**P**$(v)$ can be convexified in $O(n)$: $T_{conv}(n) \in O(n)$

$T(n) = O(n) + T(n-1) + O(1) \implies T(n) \in O(n^2)$

**Contracted vertices**

"Suitably placed inside **P**":

- where?
- how do they move?

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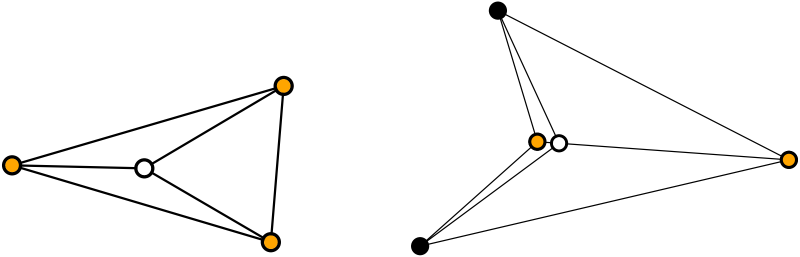
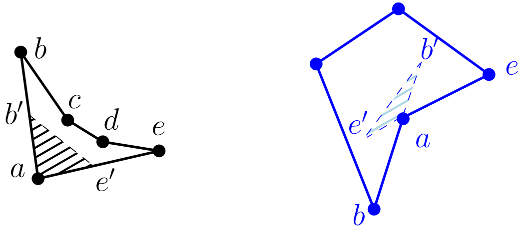
# Moving contracted vertices

Contracted degree-3 and degree-4 vertices are expressed as convex combination of their kernel-neighbors



# Moving degree-5 contracted vertices

As long as the convexity of the polygon does not change...




When it changes...

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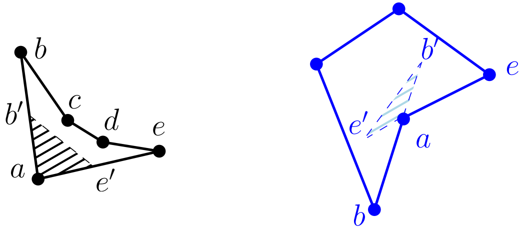
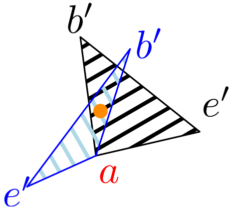



When it changes...



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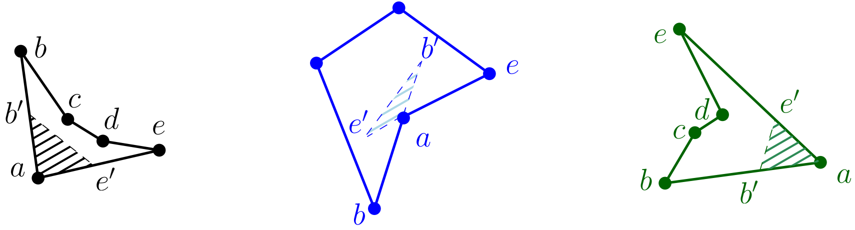
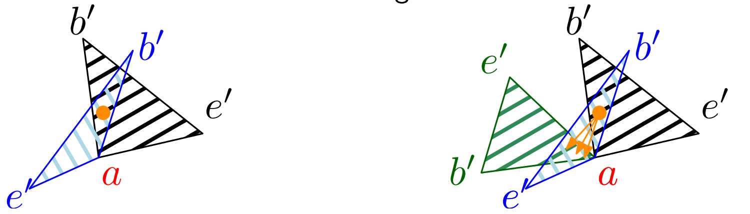



When it changes...



# Future work

## Open problems

- more efficient algorithms for general planar graphs?
  - how to compute a morphing without augmenting the drawings to represent a triangulation?
- lower bound?
- planar 3-trees admit morphing in $O(n)$, cycles in $O(n^2)$:
  - any other meaningful subclasses admitting "short" morphings?

## What we did in the meanwhile

*Unidirectional morphings* to move contracted vertices

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## Thank you!