Free-Variable Tableaux for Constant-Domain Quantified Modal Logics with Rigid and Non-Rigid Designation

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Abstract. This paper presents a sound and complete free-variable tableau calculus for constant-domain quantified modal logics, with a propositional *analytical* basis, i.e. one of the systems **K**, **D**, **T**, **K4**, **S4**. The calculus is obtained by addition of the classical free-variable γ-rule and the "liberalized" δ^{+}-rule [14] to a standard set of propositional rules. Thus, the proposed system characterizes proof-theoretically the constant-domain semantics, which cannot be captured by "standard" (non-prefixed, non-annotated) ground tableau calculi. The calculi are extended so as to deal also with non-rigid designation, by means of a simple numerical annotation on functional symbols, conveying some semantical information about the worlds where they are meant to be interpreted.

1 Introduction

Quantified modal logic (QML) can be given a model-theoretical characterization by extending the propositional Kripke semantics: a first-order modal structure is a set of first-order classical interpretations (the "possible worlds"), connected by a binary relation (the accessibility relation). However, things are not just so simple, and several issues have to be addressed (see for example [12] for an overview). Among them, possible restrictions on the designation of terms and the object domains associated to the possible worlds distinguish different "variants" of QML. When the interpretation of a ground term is required to be the same in every world, then it is said to be a *rigid* term, otherwise it is *non-rigid*. Rigid and non-rigid designation can in principle coexists within the same logic, whenever some symbols are given a rigid interpretation and others are not. On the contrary, requirements about possible relations between the universes of different worlds necessarily characterize different logics, the same way as restrictions on the accessibility relation do, on the propositional side. The most commonly considered variants of QML, in this respect, are the *constant-domain* variant, where the object domain is the same for all worlds, and the *cumulative-domain* (or *increasing-domain*) variant, where the object domains can vary, but monotonically, i.e. if w' is accessible from w, then the object domain of w is included in the domain of w'; when the domains of different worlds are independent one of the other, then domains are said to be *varying*.

Cumulative-domain QML, with rigid designation only, can easily be given proof theoretic characterizations, obtained by addition of the principles of classical first-order logic to a propositional modal system. In fact, in such systems the converse of the Barcan Formula, that characterizes cumulative domains, is provable: $\Box\forall xA \rightarrow \forall x\Box A$. Rigid designation, on the other hand, is a consequence of the instantiation rule of classical logic. Quantified modal logics with cumulative domains and rigid designation have been given sequent and tableau calculi [9, 10, 15], natural deduction proof systems [3], matrix proof procedures [21], resolution style calculi [1, 6], and have been treated by means of translation methods [2, 19].

Constant-domain logics with rigid designation can be treated axiomatically, by addition of the Barcan formula, $\forall x\Box A \rightarrow \Box\forall xA$, to the axioms and rules of classical first-order logic and modal propositional logics. Beyond the axiomatic approach, translation methods are general enough to treat constant-domain logics, and rigid as well as non-rigid designation (see, for instance, [2, 19]). Constant-domain logics with rigid designation have been formalized in the tableau style, but with the addition of prefixes labelling tableau nodes [9], as well as by means of matrix proof methods [21]: in both kinds of calculi in fact it is possible to analyse more than one possible world at a time, and this allows the proof to "go back and forth" (the same mechanism solves a similar problem raised by symmetric logics). In [9] all the variants of QML concerning the object domains of possible worlds are treated by prefixed tableau methods. A different approach is represented by modal display calculi, where the addition of the classical sequent rules for quantifiers captures constant-domain logics [4], and constant, increasing and decreasing-domain modal logics can all be presented as cut-free display sequent calculi, by use of structure-dependent rules for quantifiers [22]. A further direct approach dealing with both varying domains and non-rigid symbols has a representative in [16], which defines a resolution method for epistemic logics, where terms can be annotated by a "bullet" constructor distinguishing rigid terms from non-rigid ones. A rather different direction is followed in [11], where the language of modal logic is enriched by means of a predicate abstraction operator, in order to capture differences on the denotation of terms, and a tableau proof procedure is presented for such a logic, with no restriction on the domains of possible worlds.

The constant-domain requirement bears some relationship with the symmetry of the accessibility relation: if a model is symmetric and its domains are cumulative, then it is a constant-domain model. Constant-domain and symmetric logics also share a proof-theoretical difficulty: it is provable that constant-domain logics cannot be captured by "standard" tableau methods, or cut-free sequent calculi. In fact, the Craig Interpolation Theorem does not hold for such logics [8], while it is a consequence, in some cases, of the existence of sound and complete cut-free Gentzen systems [9]:

> It follows that there can be no "reasonable" cut-free tableau system for such logics for if there were we could use it, or the related symmetric Gentzen system, to prove an Interpolation result ([9], p.383).

This paper shows that, however, constant-domain logics – with a propositional *analytical* basis, i.e. one of the systems **K**, **D**, **T**, **K4**, **S4**, whose Gentzen calculi enjoy the cut-elimination property – can be given a simple proof-theoretical characterization by means of free-variable tableau systems (obviously, the proof of the Interpolation Theorem given in [9] for cumulative-domain QML does not extend to such systems. See also Section 5). Moreover, non-rigid designation can also be treated, by means of a simple numerical annotation on functional symbols, conveying some semantical information about the worlds where they are meant to be interpreted. A similar annotation mechanism is used in the tableau calculi for cumulative and varying domain QML, with either rigid or non-rigid designation, presented in [7].

The paper is organized as follows: Section 2 gives an informal presentation of the free-variable calculus for constant-domain logics with rigid designation only, and the main intuitions justifying its appropriateness. The syntax and formal semantics of constant-domain logics with both rigid and non-rigid designation are presented in Section 3, and the tableau systems for such logics in Section 4, where the main lines of the soundness and completeness proofs are also given. Section 5 concludes this work.

2 The role of the δ-rule in modal calculi

Ground tableaux systems for QML with increasing domains and rigid designation are easily obtained by addition of the classical quantifier rules to the modal systems [9]:

$$(\gamma_G)\ \frac{\forall x A, S}{A[t/x], \forall x A, S} \qquad\qquad (\delta_G)\ \frac{\exists x A, S}{A[c/x], S}$$

In the γ_G-rule, t is any ground term occurring in the branch. In the δ_G-rule, c is a new constant, that does not occur in $\{\exists x A\} \cup S$.

The ground approach cannot be adapted to the constant-domain case. A naive and intuitive account of the difficulties that are encountered can be given as follows. In order to be complete, the expansions of a universal formula $\forall x A$ should include the instances $A[c/x]$ where c is a constant denoting an object known to exist in a further accessible world. For instance, if we attempt to prove the Barcan formula (with the π-rule for system **K**):

$$\cfrac{\cfrac{\cfrac{\cfrac{\neg(\forall x \Box A \rightarrow \Box \forall x A)}{\forall x \Box A,\ \neg \Box \forall x A}\ (\alpha)}{\Box A[a/x],\ \forall x \Box A,\ \neg \Box \forall x A}\ (\gamma_G)}{A[a/x],\ \neg \forall x A}\ (\pi_K)}{A[a/x],\ \neg A[c/x]}\ (\delta_G)$$

we observe that, in order to obtain a closed tableau, we need to get a leaf containing also $A[c/x]$. But, after the application of the π-rule, the universal formula is not available any more. So, in principle, the formula $\forall x \Box A$ should be expanded to $\Box A[c/x]$, before the application of the π-rule. This is reasonable, since the object denoted by c, that will further on be known to exist, belongs to the domain of the presently considered world, too. But, if we change the tableau, replacing every occurrence of the constant a with c, then the constant c cannot be used any more in the application of the δ_G-rule, since it is not "new" to the node.

Free-variable tableau calculi for QML with increasing domains and rigid designation can be obtained in the same straightforward manner as the ground calculi are, by addition of the classical free-variable rules, and the substitution rule, to a modal propositional system [10]. The quantifier rules are:

$$(\gamma)\ \frac{\forall x A, S}{A[v/x], \forall x A, S} \qquad (\delta)\ \frac{\exists x A, S}{A[f(v_1, ..., v_k)/x], S}$$

In the γ-rule, v is a new free variable (also called *parameter*). In the δ-rule, f is a new functional symbol, and $v_1, ..., v_k$ are all the parameters occurring in $\{\exists x A\} \cup S$.

It has been proved [14] that the classical calculus stays sound if, in the δ-rule, only the parameters actually occurring in $\exists x A$ are required to be the arguments of the Skolem function. Such a "liberalized" rule is called the δ^+-rule. Although in the classical case the δ^+-rule is a more efficient - but equivalent - reformulation of the δ-rule, it is not sound with respect to the cumulative-domain variant of QML. In fact, the following tableau shows that the δ^+-rule makes the Barcan formula provable in a modal calculus (with the π-rule for system **K**):

$$\cfrac{\cfrac{\cfrac{\cfrac{\neg(\forall x \Box A \rightarrow \Box \forall x A)}{\forall x \Box A,\ \neg\Box\forall x A}\ (\alpha)}{\Box A[v/x],\ \forall x \Box A,\ \neg\Box\forall x A}\ (\gamma)}{A[v/x],\ \neg\forall x A}\ (\pi_K)}{A[v/x],\ \neg A[c/x]}\ (\delta^+)$$

The tableau above is closed by application of the substitution $\{c/v\}$.

In this paper we show that the free-variable modal tableau calculi obtained by replacing the δ-rule with the δ^+-rule are sound and complete with respect to the constant-domain variants of QML, on a propositional basis that is one of the analytical systems **K**, **D**, **T**, **K4** or **S4**. Note, however, that this fact does not imply that Skolemization preserves satisfiability in constant-domain QML, because quantifiers cannot always cross the modal operators. In fact, $\forall x \Diamond A \rightarrow \Diamond \forall x A$ is not valid, even in constant-domain QML. As a consequence, although run-time Skolemization is allowed, formulae cannot be initially skolemized.

Intuitively, the shift from cumulative to constant domains caused by the "liberalization" on the arguments of Skolem functions corresponding to the δ^+-rule is due to the following reasons. The key remark is that the role of the parameters $v_1, ..., v_k$ in the new Skolem term $f(v_1, ..., v_k)$, introduced by the δ-rule in free-variable calculi, is to prevent the unification of such a term with any of $v_1, ..., v_k$. Thus, the effect of reducing the set of parameters in Skolem terms is to make more unifications possible, in particular also the unification of a parameter v with a Skolem term that still has to be introduced when v is generated.

Clearly, in order to preserve soundness, the unification of a parameter v with a term $f(v_1, ..., v_k)$ must be forbidden when the existential quantifier "generating" $f(v_1, ..., v_k)$ is in the scope of the universal quantifier generating v, otherwise the dependence $\forall$-$\exists$ is lost. As a consequence, in that case v must be one of $v_1, ..., v_k$, so that v and $f(v_1, ..., v_k)$ are not unifiable. With this exception, v and $f(v_1, ..., v_k)$ can be unified: if the two terms are introduced "in the same world", the reason is the same as in the classical case. Also, if v ranges on the domain of a previous world w, but the corresponding quantifier does not dominate the existential quantifier corresponding to the Skolem term, then it can be unified with such a term, because, in the constant-domain variant of QML, the object existing in any world accessible from w belongs to the domain of w too.

On the contrary, in the free-variable calculus for cumulative-domain logics the parameters occurring in a Skolem term $f(v_1, ..., v_k)$ must include all the parameters introduced in "previous" worlds, because a universal quantifier varying on the domain of a world w must not be instantiated with a term denoting an object possibly belonging only to the domain of another world (accessible from w). Cumulative-domain QML can be given a free-variable tableau calculus with the liberalized δ^+-rule, but only with the addition of numerical annotations on functional symbols, in the style of [7]. In fact, in that case, the unification of a parameter related to a world w with a term that is not guaranteed to belong to the domain of w is prevented by a suitable restriction on the notion of substitution, that takes into account symbol annotations.

The tableau system for constant-domain QML considered above is a special case of the calculus presented in detail in Section 4, where both rigid and non-rigid designation are allowed.

3 Constant-Domain QML with Rigid and Non-Rigid Terms

A first order modal language L is constituted by logical symbols (propositional connectives, quantifiers, modal operators and a countable set X of individual variables), a non empty set L_P of predicate symbols, and a set L_F of functional symbols with an associated arity. The set L_F is partitioned into a set L_{F_R} of rigid functional symbols, and a disjoint set $L_{F_{NR}}$ of non-rigid functional symbols. The set L_F is the union of L_{F_R} and $L_{F_{NR}}$. Constants are considered as functional symbols with null arity. Terms are built by use of symbols from L_F and X in the usual way. We consider modal formulae in negation normal form, i.e. built out of literals (atoms and negated atoms) by use of $\wedge$, $\vee$, $\Box$, $\Diamond$ and the quantifiers $\forall$ and $\exists$. Negation over non-atomic formulae and implication are considered as defined symbols.

In the case of constant domain QML, a first-order modal interpretation $\mathcal{M}$ of a language L is a tuple $\langle W, w_0, R, D, \phi, \pi \rangle$ such that:

- W is a non empty set (the set of "possible worlds");
- w_0 is a distinguished element of W (the "initial world");[1]
- R is a binary relation on W (the *accessibility relation*); wRw' abbreviates $\langle w, w' \rangle \in R$;
- D is a non empty set (the object domain);
- ϕ represents the interpretation of constants and functional symbols in the language: for every world $w \in W$ and k-ary functional symbol $f \in L_F$ (with $k \geq 0$),
$$\phi(w, f) \in D^k \to D$$
Moreover, if $f \in L_{F_R}$, then for all $w, w' \in W$, $\phi(w, f) = \phi(w', f)$.
- π is the interpretation of predicate symbols: if p is a k-ary predicate symbol and $w \in W$, then $\pi(w, p) \subseteq D^k$ is a set of k-tuples of elements in D.

The interpretation function ϕ is extended to terms in the usual way, and, by an abuse of notation, $\phi(w, t)$ denotes the interpretation of t in w.

If $\mathcal{M} = \langle W, w_0, R, D, \phi, \pi \rangle$ is an interpretation of the language L, the *language of the model* $\mathcal{M}$, $L(D)$, is obtained from L by addition of a "name" for each $d \in D$, i.e. a new constant $\overline{d}$. It is assumed that for every $d \in D$ and $w \in W$, $\phi(w, \overline{d}) = d$. Note that the interpretation of names $\overline{d}$ is always rigid, i.e. $\overline{d} \in L(D)_{F_R}$

The relation $\models$ between an interpretation $\mathcal{M} = \langle W, w_0, R, D, \phi, \pi \rangle$, a world $w \in W$ and a closed formula in $L(D)$ is defined inductively as follows:

1. $\mathcal{M}, w \models p(t_1, ..., t_n)$ iff $\langle \phi(w, t_1), ..., \phi(w, t_n) \rangle \in \pi(w, p)$.
2. $\mathcal{M}, w \models \neg A$ iff $M, w \not\models A$.
3. $\mathcal{M}, w \models A \wedge B$ iff $\mathcal{M}, w \models A$ and $\mathcal{M}, w \models B$.
4. $\mathcal{M}, w \models A \vee B$ iff $\mathcal{M}, w \models A$ or $\mathcal{M}, w \models B$.
5. $\mathcal{M}, w \models \forall x A$ iff for all $d \in D$, $\mathcal{M}, w \models A[\overline{d}/x]$
6. $\mathcal{M}, w \models \exists x A$ iff there exists $d \in D$ such that $\mathcal{M}, w \models A[\overline{d}/x]$
7. $\mathcal{M}, w \models \Box A$ iff for all $w' \in W$ such that wRw', $\mathcal{M}, w' \models A$
8. $\mathcal{M}, w \models \Diamond A$ iff there is a $w' \in W$ such that wRw' and $\mathcal{M}, w' \models A$

A closed formula A is true in $\mathcal{M}$ iff $\mathcal{M}, w_0 \models A$, and it is valid iff it is true in all interpretations.

The accessibility relation R of a modal structure can be required to satisfy additional properties, characterizing different logics: we consider seriality (**D**), reflexivity (**T**), transitivity (**K4**), both reflexivity and transitivity (**S4**). When no additional assumption on R is made, the logic is **K**.

4 The Free-Variable Tableau System

The language of any tableau for a set of formulae in the language L extends L with a denumerable set of parameters (or free variables) and a denumerable set

[1] The semantics with a distinguished initial world dates back to [17]. Obviously, the notion of validity (truth in every model) coincides with the semantics where no initial world is singled out.

of Skolem functional symbols. Moreover it contains annotated symbols f^n, for every $f \in L_{F_{NR}}$ and $n \in \mathbb{N}$.

Definition 1. *Let L be a modal language, $V = \{v_1, v_2, ...\}$ an infinite denumerable set of new symbols, called free variables or parameters, and F^{Sk} a denumerable set of new function symbols such that, for each $k \in \mathbb{N}$, F^{Sk} contains infinitely many function symbols of arity k. Then the* labelled free-variable extension *of L is the language L^* such that:*

$$L_P^* = L_P$$
$$L_{F_R}^* = L_{F_R} \cup F^{Sk} \cup \{f^n \mid f \in L_{F_{NR}}, n \in \mathbb{N}\}$$
$$L_{F_{NR}}^* = L_{F_{NR}}$$

Terms in L^* are built up by use of L_F^*, X and V in the usual way.

Note that, in the definition above, Skolem functions and annotated functional symbols are considered as rigid symbols.

Definition 2.

1. *A symbol occurrence in a formula or set of formulae is called a non-modal occurrence if it is in the scope of no modal operators.*
2. *An n-annotated modal formula is a modal formula where non-modal occurrences of non-rigid functional symbols are all annotated with n. A formula is annotated if it is n-annotated for some n.*
3. *A term t is completely annotated iff every non-rigid functional symbol in t is annotated.*
4. *A modal substitution σ is a function from the set of free-variables V to completely annotated terms in L^*, that is the identity almost everywhere. Substitutions are denoted as usual by expressions of the form $\{t_1/v_1, ..., t_m/v_m\}$. Unifiers, i.e. solutions of unification problems, and most general unifiers (m.g.u.) are defined as in the classical case.*
5. *If A is a modal formula and $n \in \mathbb{N}$, then A^n is obtained from A by annotating each non-modal occurrence of a non-rigid functional symbol with n. If S is a set of modal formulae, then $S^n = \{A^n \mid A \in S\}$.*
   For instance, if $f \in L_{F_R}$, $c, g \in L_{F_{NR}}$, and $A = \forall x(p(f(x,c), g(x)) \land \Box q(g(c)))$, then $A^n = \forall x(p(f(x,c^n), g^n(x)) \land \Box q(g(c)))$.
6. *If S is a set of modal formulae and $n \in \mathbb{N}$, then the single node S^n is an initial tableau for S.*

The main intuition behind the annotation of non-rigid symbols is that, since a symbol in $L_{F_{NR}}$ can occur with different annotations in a tableau branch, its annotations distinguish the designations of such a symbol in the different worlds corresponding to the tableau nodes. Non-modal occurrences of non-rigid symbols are initialized with a given annotation n (for instance 0), that identifies the initial world of the searched model. Symbols of $L_{F_{NR}}$ occurring in the scope of a modal operator receive their annotations only when, by application of a modal rule, they come to the surface (by means of operation 5 in Definition 2).

Classical propositional rules	
$(\alpha)\quad \dfrac{A\wedge B, S}{A, B, S}$	$(\beta)\quad \dfrac{A\vee B, S}{A, S \qquad B, S}$
ν-rules	
$(\nu_D)\quad \dfrac{\Box S, S'}{S^m}$ where $m\in\mathbb{N}$ is new in the whole tableau	$(\nu_T)\quad \dfrac{\Box A, S}{A^n, \Box A, S}$ where $n\in\mathbb{N}$ either occurs as an annotation of some symbol in the premisse or it is new in the whole tableau
π-rules	
$(\pi_K)\quad \dfrac{\Diamond A, \Box S, S'}{A^m, S^m}$ where $m\in\mathbb{N}$ is new in the whole tableau	$(\pi_4)\quad \dfrac{\Diamond A, \Box S, S'}{A^m, S^m, \Box S}$ where $m\in\mathbb{N}$ is new in the whole tableau

Fig. 1. Propositional expansion rules

We consider a simple set of propositional tableau rules (others may be chosen as well), shown in Figure 1, where S, S' are sets of modal formulae (in L^*), $\Box S$ stands for $\{\Box A \mid A\in S\}$, S' is a set of non-boxed modal formulae, comma is set union. The propositional part of the tableau systems we consider consists of the classical rules, and: the rule π_K in the systems **K**, **D** and **T**, the rule π_4 in **K4** and **S4**, the rule ν_T in **T** and **S4**, the rule ν_D in **D**. The *quantifier expansion rules* are:

Quantifier rules	
$(\gamma)\quad \dfrac{\forall x A, S}{A[v/x], \forall x A, S}$ where v is a new parameter	$(\delta^+)\quad \dfrac{\exists x A, S}{A[f(v_1,...,v_k)/x], S}$ where f is a new Skolem function, i.e. a symbol in F^{Sk} that does not occur elsewhere in the tableau, and $v_1,...,v_k$ are all the parameters occurring in $\exists x A$

Adopting a terminology from [13], the rules ν_D, π_K and π_4 are called *dynamic*, the others are called *static*. So, a non-annotated non-rigid symbol can get an annotation only with the application of dynamic rules. Note, moreover, that only non-modal occurrences of non-rigid functional symbols in a tableau node can be annotated. As a consequence, every tableau node is a set of n-annotated formulae, for some $n \in \mathbb{N}$. In fact, the expansions obtained by application of a dynamic rule never contain symbols with different annotations (i.e. if both f^n and g^k occur in the expansion, then $n = k$). Such a property is preserved by applications of the ν_T-rule. Therefore, no tableau node contains symbols with different annotations.

The substitution rule we adopt here requires the following preliminary definitions:

- If P and Q are two atomic formulae in L^*, then $P = Q$ if P and Q are identical, including their annotations.
- Let $S_1, ..., S_k$ be all the leaves of a free-variable tableau $\mathcal{T}$, and, for each $i = 1, ..., k$, let P_i and $\neg Q_i$ be literals in S_i. If the modal substitution σ is a solution of the unification problem $P_1 = Q_1, ..., P_k = Q_k$, then σ is an *atomic closure substitution for* $\mathcal{T}$. If it is a most general solution for such a unification problem, then it is a *most general atomic closure substitution for* $\mathcal{T}$.

The substitution rule is then the following:

Most General Atomic Closure Substitution Rule
If $\mathcal{T}$ is a tableau for a set S of sentences in L and the modal substitution σ is a most general atomic closure substitution for $\mathcal{T}$, then the tree $\mathcal{T}\sigma$, obtained by applying σ to $\mathcal{T}$, is a tableau for S.

A tableau is closed iff each of its leaves contains a pair of complementary literals, i.e. literals P and $\neg P$. A closed tableau for a formula $\neg A$ is a tableau proof of A, and a closed tableau for a set of formulae S is a refutation of S.

As a first example, consider the language L with the unary predicates p and q, the rigid constant c and the non-rigid unary functional symbol f. The following tableau shows that $\{\Box p(f(c)), \forall x \Box(p(x) \to q(x)), \neg\Box q(f(c))\}$ is refutable in $\mathbf{K}$, since it can be closed by the substitution $\{f^0(c)/v\}$:

$$
\cfrac{\cfrac{\cfrac{\Box p(f(c)), \forall x \Box(p(x) \to q(x)), \neg\Box q(f(c))}{\Box p(f(c)), \Box(p(v) \to q(v)), \forall x \Box(p(x) \to q(x)), \neg\Box q(f(c))}\ (\gamma)}{p(f^0(c)), p(v) \to q(v), \neg q(f^0(c))}\ (\pi_K)}{p(f^0(c)), \neg p(v), \neg q(f^0(c)) \qquad p(f^0(c)), q(v), \neg q(f^0(c))}\ (\beta)
$$

And now a tableau that should not and indeed does not close, thanks to the different annotations of f in different worlds:

$$
\cfrac{\cfrac{p(f^0(c)),\ \forall x(p(x) \to \Box q(x)),\ \neg\Box q(f(c))}{p(f^0(c)),\ p(v) \to \Box q(v),\ \forall x(p(x) \to \Box q(x)),\ \neg\Box q(f(c))}\ (\gamma)}{\begin{array}{c} p(f^0(c)),\ \neg p(v), \\ \forall x(p(x) \to \Box q(x)), \\ \neg\Box q(f(c)) \end{array} \qquad \cfrac{\begin{array}{c} p(f^0(c)),\ \Box q(v), \\ \forall x(p(x) \to \Box q(x)),\ \neg\Box q(f(c)) \end{array}}{q(v),\ \neg q(f^1(c))}\ (\pi_k)}\ (\beta)
$$

The tableau above is in fact a failed attempt to give a refutation of the satisfiable set $S = \{p(f(c)), \forall x(p(x) \rightarrow \Box q(x)), \neg\Box q(f(c))\}$, where c is rigid and f is non rigid, in any of the systems **K**, **D** or **T**. The tableau cannot be closed, since the unification problem $p(f^0(c)) = p(v),\ q(v) = q(f^1(c))$ has no solution (note that S is satisfiable because f is non-rigid). And no matter how many applications of the γ-rule are added to the tableau above, the set S cannot be refuted.

Note that, in the absence of non-rigid symbols, i.e. when $L_{F_{NR}} = \emptyset$, the calculus presented in this sections results exactly from the addition of the classical free-variable γ and δ^+-rules to a standard tableau calculus for any of the considered propositional modal logics.

4.1 Soundness and Completeness of the Free-Variable Calculus

In this section we give a sketchy account of the soundness and completeness proofs. The proofs make use of a modal Substitution Theorem, stating that, if $\mathcal{M} = \langle W, w_0, R, D, \phi, \pi, \rangle$ is an interpretation of a modal language L, $w \in W$, A is a formula with only the variable x free, and t, t' are ground terms of L, then:

1. if the interpretations of t and t' are rigid and equal, i.e. for all $w' \in W$, $\phi(w', t) = \phi(w', t')$, then $\mathcal{M}, w \models A[t/x]$ iff $\mathcal{M}, w \models A[t'/x]$.
2. If x does not occur in A in the scope of any modal operator and $\phi(w, t) = \phi(w, t')$, then $\mathcal{M}, w \models A[t/x]$ iff $\mathcal{M}, w \models A[t'/x]$.

Theorem 1 (Soundness). *If there is a closed tableau for S, then S is unsatisfiable.*

The soundness proof runs along standard lines, except for the fact that the following notion of tableau satisfiability must be considered:

Definition 3. *Let $\mathcal{T}$ be a tableau, with parameters among $v_1, ..., v_k$, and let $\mathcal{M} = \langle W, w_0, R, D, \phi, \pi\rangle$ be a modal interpretation. Then $\mathcal{M} \models \mathcal{T}$ iff for all $d_1, ..., d_k \in D$ there is a leaf S of $\mathcal{T}$ and a world $w \in W$ such that*

$$\mathcal{M}, w \models S\{\overline{d_1}/v_1, ..., \overline{d_k}/v_k\}$$

$\mathcal{T}$ is satisfiable iff $\mathcal{M} \models \mathcal{T}$ for some interpretation $\mathcal{M}$.

Soundness follows from the fact that, if S is a satisfiable set of formulae, then any tableau for S is satisfiable. The proof of this fact is an induction on tableaux. We just note here that it is the case of the substitution rule that makes essential use of the hypothesis that the object domain is the same for each world.

Theorem 2 (Completeness). *If S is an unsatisfiable set of modal formulae, then there exists a closed tableau for S.*

The completeness proof also follows the standard approach consisting of the construction of a canonical model of a set of formulae having no closed tableau. In general, we deal with possibly infinite sets of formulae, and make use of the following notion of *tab-consistency*: S is tab-consistent iff for every finite subset $S' \subseteq S$ there is no closed tableau for S'.

The reformulation of the basic notion of downward saturated sets of formulae is the following: a (finite or infinite) set S of n-annotated modal formulae, possibly containing parameters, is n-downward saturated iff

1. S does not contain any pair of literals P_1 and $\neg P_2$ such that P_1 and P_2 are unifiable;
2. if $A \wedge B \in S$ then $A \in S$ and $B \in S$
3. if $A \vee B \in S$ then either $A \in S$ or $B \in S$;
4. if $\forall x A \in S$ then $A[v/x] \in S$ for every parameter $v \in V$;
5. if $\exists x A \in S$, and $v_1, ..., v_k$ are all the parameters occurring in A, then for some Skolem functional symbol f, $A[f(v_1, ..., v_k)/x] \in S$;
6. if the logic is either **T** or **S4** and $\Box A \in S$ then $A^n \in S$.

It can easily be proved that if a set S of n-annotated modal formulae is tab-consistent, then:

1. If $A \wedge B \in S$, then $S \cup \{A, B\}$ is tab-consistent.
2. If $A \vee B \in S$, then either $S \cup \{A\}$ or $S \cup \{B\}$ is tab-consistent.
3. If $\forall x A \in S$ then
$$S \cup \bigcup_{v \in V} \{A[v/x]\}$$
is tab-consistent.
4. If $\exists x A \in S$, f is a functional symbol that does not occur in S and $v_1, ..., v_n$ are all the parameters occurring in A, then $S \cup \{A[f(v_1, ..., v_n)/x]\}$ is tab-consistent.
5. If the logic is either **T** or **S4** and $\Box A \in S$, then $S \cup \{A^n\}$ is tab-consistent.

The result stated above is used to prove the following:

Lemma 1. *Let L be a modal language, $n \in \mathbb{N}$ and S a (finite or infinite) set of n-annotated modal formulae in L^*. If H is a set containing an infinite number of functional symbols (for each arity) that do not occur in S and S is tab-consistent, then there exists a tab-consistent set $S^\infty \supseteq S$ of n-annotated formulae (possibly containing also symbols from H), such that S^∞ is n-downward saturated.*

The main idea behind the proof of the lemma above is standard and consists of the construction of a (pseudo-)tableau $\mathcal{T}^\infty$, rooted at S, in such a way that $\mathcal{T}^\infty$ is closed with respect to the application of static expansion rules. Such a construction uses Skolem functions in H in the applications of the δ^+-rule. The tree $\mathcal{T}^\infty$ is not properly a tableau, since its nodes may be infinite sets of formulae. In presence of such infinite objects, some care has to be taken in defining a "fair" rule application strategy.

The canonical model of a tab-consistent set of formulae S in L^* is built along standard lines, using Lemma 1. The initial world w_0 is a tab-consistent and 0-downward saturated superset of S^0, and each world w in the model is a tab-consistent and n-downward saturated set of n-annotated formulae, for some $n \in \mathbb{N}$ that is uniquely associated to w (and is called its "name"): $name(w) = n$. To each world w is also associated a set of Skolem functions $H(w)$ to be used in the application of Lemma 1, in such a way that if $w \neq w'$ then $H(w)$ and $H(w')$ are disjoint. The interpretation of the language is "syntactical": its domain D is the set of completely annotated ground terms in L^*, and the interpretation of terms is "almost" Herbrand-like: for all $w \in W$

– if $f \in L_{F_R} \cup F^{Sk}$ is either a rigid symbol in L or a Skolem function, then for all $t_1, ..., t_k \in D$: $\phi(w, f)(t_1, ..., t_k) = f(t_1, ..., t_k)$
– If $f \in L_{F_{NR}}$ is a non-rigid symbol in L, then for all $t_1, ..., t_k \in D$:
for all $m \in \mathbb{N}$: $\phi(w, f^m)(t_1, ..., t_k) = f^m(t_1, ..., t_k)$
and, if $name(w) = n$: $\phi(w, f) = \phi(w, f^n)$

So, the interpretation of annotated symbols is always rigid and, for all $w \in W$ and completely annotated ground term t in L^*, $\phi(w, t) = t$; i.e. if $t \in D$ then $\phi(w, t) = t$. In particular, the interpretation of completely annotated terms is rigid.

Finally, let σ be any surjective function from the set of parameters V to the set of completely annotated ground terms of the extended language: for each completely annotated ground term t there exists $v \in V$ such that $\sigma(v) = t$. If A is a formula, then $\sigma(A)$ denotes the formula obtained from A by replacement of each parameter v in A by $\sigma(v)$. If S is a set of formulae, $\sigma(S) = \{\sigma(A) \mid A \in S\}$. The interpretation function π is then defined as follows: for all $w \in W$ and k-ary predicate symbols p in L^*:

$$\pi(w, p) = \{\langle t_1, ..., t_k \rangle \mid p(t_1, ..., t_k) \in \sigma(w)\}$$

It can be proved that, for all $w \in W$ and for every formula A:

$$\text{if } A \in w \text{ then } \mathcal{M}, w \models \sigma(A)$$

Hence, in particular, $\mathcal{M}, w_0 \models \sigma(w_0)$ and, since S^0 contains no parameters and $S^0 \subseteq w_0$, $\mathcal{M}, w_0 \models S^0$. This easily implies that $\mathcal{M}, w_0 \models S$.

5 Concluding Remarks

This work shows that the addition of the free-variable γ-rule and δ^+-rule, rather than the more constraining δ-rule, to standard propositional modal tableau calculi captures proof-theoretically the constant domain semantics for QML in the case of rigid designation.[2] Moreover, non-rigid designation can also be answered for, by means of simple annotations on non-rigid functional symbols. Thus, it makes it apparent the sensitivity of modal tableau free-variable calculi to alternative formulations of the δ-rule, contrarily to the classical case, and similarly to what happens for other non classical logics. For instance, in [5], where a free-variable sequent system which is sound and complete for linear logic is proposed, it is shown that an analogous liberalization of the $\forall_R$ and the $\exists_L$ rules is unsound: in that case, too, restricting the set of arguments of Skolem functions results in a shift to a different "logic". [20] and [18] make a fine analysis of the relationship between the arguments of Skolem functions in run-time Skolemization and rule permutability in cut-free sequent calculi for intuitionist and linear logic, respectively.

[2] Although complete proofs have been carried out only for logics whose propositional basis is among **K**, **D**, **T**, **K4**, **S4**, the extension to other cut-free propositional calculi should be straightforward.

As already observed, even in the case of rigid designation, "standard" (i.e. non-annotated, non-prefixed) *ground* tableau systems for constant-domain QML cannot be given, as a consequence of the failure of Craig's Interpolation Theorem [8, 9]. However, the use of unification in the quantifier rules, which is the essential feature of free-variable tableaux, makes it possible to give a proof-theoretical characterization of constant domains in the Fitting-Gentzen style.

In Figure 2, some of the inference rules of a sequent calculus for QML which is sound and complete with respect to constant domains and rigid designation are provided. This calculus is formulated in a completely standard Gentzen style, but for the fact that unification and run-time Skolemization are embedded in the quantifier rules. A *proof* of a sequent $\mathcal{S}$ in such a formal system is given by a deduction tree $\mathcal{T}$, whose root is $\mathcal{S}$, and *a substitution* σ such that all the leaves of $\mathcal{T}\sigma$ are axioms, i.e. sequents of the form $S, A \vdash A, S'$. For the sake of concision, we present here only the calculus for system **T**.

$\dfrac{S \vdash S', A}{S, \neg A \vdash S'}\ (\neg_L)$	$\dfrac{S, A \vdash S'}{S \vdash S', \neg A}\ (\neg_R)$
$\dfrac{S, A \vdash S' \qquad S, B \vdash S'}{S, A \vee B \vdash S'}\ (\vee_L)$	$\dfrac{S \vdash S', A, B}{S \vdash S', A \vee B}\ (\vee_R)$
$\dfrac{S, \forall x A, A[v/x] \vdash S'}{S, \forall x A \vdash S'}\ (\forall_L)$ where v is a new parameter	$\dfrac{S \vdash S', A[f(v_1, ..., v_n)/x]}{S \vdash \forall x A}\ (\forall_R)$ where $v_1, ..., v_n$ are all the parameters in A
$\dfrac{S, A[f(v_1, ..., v_n)/x] \vdash S'}{S, \exists x A \vdash S'}\ (\exists_L)$ where $v_1, ..., v_n$ are all the parameters in A	$\dfrac{S \vdash S', \exists x A, A[v/x]}{S \vdash S', \exists x A}\ (\exists_R)$ where v is a new parameter
$\dfrac{S, A, \Box A \vdash S'}{S, \Box A \vdash S'}\ (\Box_L)$	$\dfrac{S \vdash S', A}{S_0, \Box S \vdash \Diamond S', \Box A, S_1}\ (\Box_R)$

Fig. 2. Rules of a Free-Variable Sequent Calculus for **T**-QML with Constant Domains and Rigid Designation

Such sequent systems can obviously be reformulated as *symmetric* calculi (in the sense of [9], i.e. calculi where formulae never cross the sequent arrow), taking negation as a defined connective and eliminating the rules for negation and implication. In this case, axioms include also sequents of the form $S, A, \neg A \vdash S'$ and $S \vdash A, \neg A, S'$. The equivalence of such symmetric calculi and the tableau calculi presented in Section 4 is immediate: if S and S' are sets of formulae in negation normal form, a sequent $S \vdash S'$ is provable in the symmetric sequent calculus if and only if the set of formulae $S \cup \{\neg A \mid A \in S'\}$ has a refutation in the (corresponding) tableau system of Section 4.

We conclude this section showing where an attempt to adapt Fitting's proof of the Interpolation Theorem for "ordinary" cut-free sequent calculi ([9]) to the calculus presented in Figure 2 fails. In fact, the rules $\exists_L$ and $\forall_R$ (corresponding to the δ^+-rule of the free variable tableau system) do not preserve the existence of interpolants. This is not surprising: such rules are not locally sound, since "liberalized Herbrandization" (the δ^+-rule) does not correspond to the *eigenvariable* condition in sequent systems, like standard Herbrandization (the δ-rule) does. As a simple example, let us consider the following deduction tree in the (symmetric) sequent calculus:

$$\dfrac{\dfrac{\dfrac{p(v) \vdash p(c)}{p(v) \vdash \forall x\, p(x)}\ (\forall_R)}{\forall x \Box p(x), \Box p(v) \vdash \Box \forall x\, p(x)}\ (\Box_R)}{\forall x \Box p(x) \vdash \Box \forall x\, p(x)}\ (\forall_L)$$

An application of the substitution $\{c/v\}$ produces a proof of the valid sequent (in **T**-QML with constant domains) $\forall x \Box p(x) \vdash \Box \forall x p(x)$ (corresponding to the Barcan formula), starting with the inference:

$$\dfrac{p(c) \vdash p(c)}{p(c) \vdash \forall x\, p(x)}\ (\forall_R)$$

Now, following the lines of Fitting's proof, we immediately find the trivial interpolant $p(c)$ for the axiom. Such a formula should be an interpolant also for the sequent $p(c) \vdash \forall x p(x)$, derived via an application of the $\forall_R$-rule (corresponding to a δ^+-rule in the tableaux systems). But this is clearly false.

References

1. M. Abadi and Z. Manna. Modal theorem proving. In *Proc. of the 8th Int. Conf. on Automated Deduction*, number 230 in LNCS, pages 172–189, Berlin, 1986. Springer.
2. Y. Auffray and P. Enjalbert. Modal theorem proving: an equational viewpoint. *Journal of Logic and Computation*, 2:247–297, 1992.
3. D. Basin, M. Matthews, and L. Viganò. Labelled modal logics: Quantifiers. *Journal of Logic Language and Information*, 7(3):237–263, 1998.
4. N. Belnap. Display logic. *J. of Philosophical Logic*, 11:375–417, 1982.

5. S. Cerrito. Herbrand methods for sequent calculi; unification in LL. In K. Apt, editor, *Proc. of the Joint International Conference and Symposium on Logic Programming*, pages 607–622, 1992.
6. M. Cialdea. Resolution for some first order modal systems. *Theoretical Computer Science*, 85:213–229, 1991.
7. M. Cialdea Mayer and S. Cerrito. Variants of first-order modal logics. In R. Dyckhoff, editor, *Proc. of Tableaux 2000*, LNCS. Springer Verlag, 2000.
8. K. Fine. Failures of the interpolation lemma in quantified modal logic. *Journal of Symbolic Logic*, 44(2), 1979.
9. M. Fitting. *Proof Methods for Modal and Intuitionistic Logics.* Reidel Publishing Company, 1983.
10. M. Fitting. First-order modal tableaux. *Journal of Automated Reasoning*, 4:191–213, 1988.
11. M. Fitting. On quantified modal logic. *Fundamenta Informaticae*, 39:105–121, 1999.
12. J. W. Garson. Quantification in modal logic. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume II, pages 249–307. D. Reidel Publ. Co., 1984.
13. R. Goré. Tableau methods for modal and temporal logics. In M. D'Agostino, G. Gabbay, R. Hähnle, and J. Posegga, editors, *Handbook of tableau methods.* Kluwer, 1999.
14. R. Hähnle and P. H. Schmitt. The liberalized δ-rule in free variable semantic tableaux. *Journal of Automated Reasoning*, 13:211–222, 1994.
15. P. Jackson and H. Reichgelt. A general proof method for first-order modal logic. In *Proc. of the 10th Joint Conf. on Artificial Intelligence (IJCAI '87)*, pages 942–944. Morgan Kaufmann, 1987.
16. K. Konolige. Resolution and quantified epistemic logics. In J. H. Siekmann, editor, *Proc. of the 8th Int. Conf. on Automated Deduction (CADE 86)*, number 230 in LNCS, pages 199–208. Springer, 1986.
17. S. A. Kripke. Semantical analysis of modal logic I: Normal propositional calculi. *Zitschrift für Mathematische Logik und Grundlagen der Mathematik*, 9:67–96, 1963.
18. P. D. Lincoln and N. Shankar. Proof search in first order linear logic and other cut-free sequent calculi. In S. Abramsky, editor, *Proc. of the Ninth Annual IEEE Symposium on Logic In Computer Science (LICS)*, pages 282–291. IEE Computer Societ Press, 1994.
19. H. J. Ohlbach. Semantics based translation methods for modal logics. *Journal of Logic and Computation*, 1(5):691–746, 1991.
20. N. Shankar. Proof search in the intuitionist sequent calculus. In D. Kapur, editor, *Proc. of the Eleventh International Conference on Automated Deduction (CADE)*, number 607 in LNCS, pages 522–536. Springer Verlag, 1992.
21. L. A. Wallen. *Automated Deduction in Nonclassical Logics: Efficient Matrix Proof Methods for Modal and Intuitionistic Logics.* MIT Press, 1990.
22. H. Wansing. Predicate logics on display. *Studia Logica*, 62(1):49–75, 1962.